3.11 Hyperbolic Functions

1. (a) $\sinh 0 = \frac{1}{2}(e^0 - e^0) = 0$ (b) $\cosh 0 = \frac{1}{2}(e^0 + e^0) = \frac{1}{2}(1+1) = 1$

2. (a) $\tanh 0 = \dfrac{(e^0 - e^{-0})/2}{(e^0 + e^{-0})/2} = 0$ (b) $\tanh 1 = \dfrac{e^1 - e^{-1}}{e^1 + e^{-1}} = \dfrac{e^2 - 1}{e^2 + 1} \approx 0.76159$

3. (a) $\sinh(\ln 2) = \dfrac{e^{\ln 2} - e^{-\ln 2}}{2} = \dfrac{e^{\ln 2} - (e^{\ln 2})^{-1}}{2} = \dfrac{2 - 2^{-1}}{2} = \dfrac{2 - \frac{1}{2}}{2} = \dfrac{3}{4}$

   (b) $\sinh 2 = \frac{1}{2}(e^2 - e^{-2}) \approx 3.62686$

4. (a) $\cosh 3 = \frac{1}{2}(e^3 + e^{-3}) \approx 10.06766$ (b) $\cosh(\ln 3) = \dfrac{e^{\ln 3} + e^{-\ln 3}}{2} = \dfrac{3 + \frac{1}{3}}{2} = \dfrac{5}{3}$

5. (a) $\operatorname{sech} 0 = \dfrac{1}{\cosh 0} = \dfrac{1}{1} = 1$ (b) $\cosh^{-1} 1 = 0$ because $\cosh 0 = 1$.

6. (a) $\sinh 1 = \frac{1}{2}(e^1 - e^{-1}) \approx 1.17520$

   (b) Using Equation 3, we have $\sinh^{-1} 1 = \ln\left(1 + \sqrt{1^2 + 1}\right) = \ln\left(1 + \sqrt{2}\right) \approx 0.88137$.

7. $\sinh(-x) = \frac{1}{2}[e^{-x} - e^{-(-x)}] = \frac{1}{2}(e^{-x} - e^x) = -\frac{1}{2}(e^{-x} - e^x) = -\sinh x$

8. $\cosh(-x) = \frac{1}{2}[e^{-x} + e^{-(-x)}] = \frac{1}{2}(e^{-x} + e^x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$

9. $\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^x) = e^x$

10. $\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^{-x}) = e^{-x}$

11.
$$\begin{aligned}\sinh x \cosh y + \cosh x \sinh y &= \left[\tfrac{1}{2}(e^x - e^{-x})\right]\left[\tfrac{1}{2}(e^y + e^{-y})\right] + \left[\tfrac{1}{2}(e^x + e^{-x})\right]\left[\tfrac{1}{2}(e^y - e^{-y})\right] \\ &= \tfrac{1}{4}[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})] \\ &= \tfrac{1}{4}(2e^{x+y} - 2e^{-x-y}) = \tfrac{1}{2}[e^{x+y} - e^{-(x+y)}] = \sinh(x+y)\end{aligned}$$

12.
$$\begin{aligned}\cosh x \cosh y + \sinh x \sinh y &= \left[\tfrac{1}{2}(e^x + e^{-x})\right]\left[\tfrac{1}{2}(e^y + e^{-y})\right] + \left[\tfrac{1}{2}(e^x - e^{-x})\right]\left[\tfrac{1}{2}(e^y - e^{-y})\right] \\ &= \tfrac{1}{4}[(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}) + (e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y})] \\ &= \tfrac{1}{4}(2e^{x+y} + 2e^{-x-y}) = \tfrac{1}{2}\left[e^{x+y} + e^{-(x+y)}\right] = \cosh(x+y)\end{aligned}$$

13. Divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$:

$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} \quad \Leftrightarrow \quad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

14.
$$\tanh(x+y) = \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y} = \frac{\dfrac{\sinh x \cosh y}{\cosh x \cosh y} + \dfrac{\cosh x \sinh y}{\cosh x \cosh y}}{\dfrac{\cosh x \cosh y}{\cosh x \cosh y} + \dfrac{\sinh x \sinh y}{\cosh x \cosh y}}$$
$$= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

15. Putting $y = x$ in the result from Exercise 11, we have

$\sinh 2x = \sinh(x + x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x$.

16. Putting $y = x$ in the result from Exercise 12, we have

$\cosh 2x = \cosh(x + x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x$.

17. $\tanh(\ln x) = \dfrac{\sinh(\ln x)}{\cosh(\ln x)} = \dfrac{(e^{\ln x} - e^{-\ln x})/2}{(e^{\ln x} + e^{-\ln x})/2} = \dfrac{x - (e^{\ln x})^{-1}}{x + (e^{\ln x})^{-1}} = \dfrac{x - x^{-1}}{x + x^{-1}} = \dfrac{x - 1/x}{x + 1/x} = \dfrac{(x^2 - 1)/x}{(x^2 + 1)/x} = \dfrac{x^2 - 1}{x^2 + 1}$

18. $\dfrac{1 + \tanh x}{1 - \tanh x} = \dfrac{1 + (\sinh x)/\cosh x}{1 - (\sinh x)/\cosh x} = \dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{\frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x})} = \dfrac{e^x}{e^{-x}} = e^{2x}$

Or: Using the results of Exercises 9 and 10, $\dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{e^x}{e^{-x}} = e^{2x}$

19. By Exercise 9, $(\cosh x + \sinh x)^n = (e^x)^n = e^{nx} = \cosh nx + \sinh nx$.

20. $\coth x = \dfrac{1}{\tanh x} \;\Rightarrow\; \coth x = \dfrac{1}{\tanh x} = \dfrac{1}{12/13} = \dfrac{13}{12}$.

$\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} \;\Rightarrow\; \operatorname{sech} x = \frac{5}{13}$ [sech, like cosh, is positive].

$\cosh x = \dfrac{1}{\operatorname{sech} x} \;\Rightarrow\; \cosh x = \dfrac{1}{5/13} = \dfrac{13}{5}$.

$\tanh x = \dfrac{\sinh x}{\cosh x} \;\Rightarrow\; \sinh x = \tanh x \cosh x \;\Rightarrow\; \sinh x = \dfrac{12}{13} \cdot \dfrac{13}{5} = \dfrac{12}{5}$.

$\operatorname{csch} x = \dfrac{1}{\sinh x} \;\Rightarrow\; \operatorname{csch} x = \dfrac{1}{12/5} = \dfrac{5}{12}$.

21. $\operatorname{sech} x = \dfrac{1}{\cosh x} \;\Rightarrow\; \operatorname{sech} x = \dfrac{1}{5/3} = \dfrac{3}{5}$.

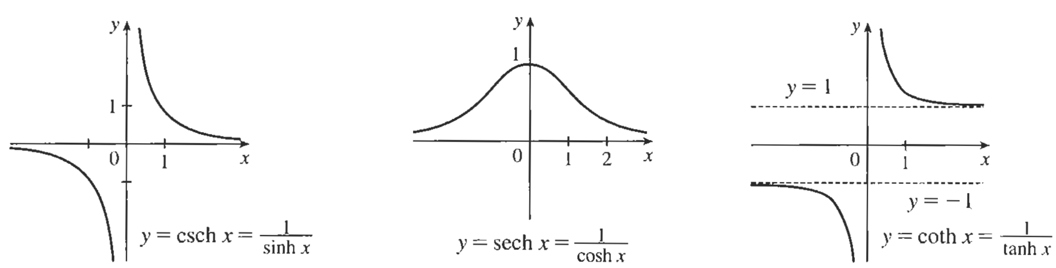
$\cosh^2 x - \sinh^2 x = 1 \;\Rightarrow\; \sinh^2 x = \cosh^2 x - 1 = \left(\frac{5}{3}\right)^2 - 1 = \frac{16}{9} \;\Rightarrow\; \sinh x = \frac{4}{3}$ [because $x > 0$].

$\operatorname{csch} x = \dfrac{1}{\sinh x} \;\Rightarrow\; \operatorname{csch} x = \dfrac{1}{4/3} = \dfrac{3}{4}$.

$\tanh x = \dfrac{\sinh x}{\cosh x} \;\Rightarrow\; \tanh x = \dfrac{4/3}{5/3} = \dfrac{4}{5}$.

$\coth x = \dfrac{1}{\tanh x} \;\Rightarrow\; \coth x = \dfrac{1}{4/5} = \dfrac{5}{4}$.

22. (a)



23. (a) $\lim\limits_{x\to\infty} \tanh x = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \lim\limits_{x\to\infty} \dfrac{1 - e^{-2x}}{1 + e^{-2x}} = \dfrac{1-0}{1+0} = 1$

(b) $\lim\limits_{x\to-\infty} \tanh x = \lim\limits_{x\to-\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \dfrac{e^x}{e^x} = \lim\limits_{x\to-\infty} \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{0-1}{0+1} = -1$

(c) $\lim\limits_{x\to\infty} \sinh x = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{2} = \infty$

(d) $\lim\limits_{x\to-\infty} \sinh x = \lim\limits_{x\to-\infty} \dfrac{e^x - e^{-x}}{2} = -\infty$

(e) $\lim\limits_{x\to\infty} \operatorname{sech} x = \lim\limits_{x\to\infty} \dfrac{2}{e^x + e^{-x}} = 0$

(f) $\lim\limits_{x\to\infty} \coth x = \lim\limits_{x\to\infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \lim\limits_{x\to\infty} \dfrac{1 + e^{-2x}}{1 - e^{-2x}} = \dfrac{1+0}{1-0} = 1$ [*Or:* Use part (a)]

(g) $\lim\limits_{x\to0^+} \coth x = \lim\limits_{x\to0^+} \dfrac{\cosh x}{\sinh x} = \infty$, since $\sinh x \to 0$ through positive values and $\cosh x \to 1$.

(h) $\lim\limits_{x\to0^-} \coth x = \lim\limits_{x\to0^-} \dfrac{\cosh x}{\sinh x} = -\infty$, since $\sinh x \to 0$ through negative values and $\cosh x \to 1$.

(i) $\lim\limits_{x\to-\infty} \operatorname{csch} x = \lim\limits_{x\to-\infty} \dfrac{2}{e^x - e^{-x}} = 0$

24. (a) $\dfrac{d}{dx}(\cosh x) = \dfrac{d}{dx}\left[\tfrac{1}{2}(e^x + e^{-x})\right] = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x$

(b) $\dfrac{d}{dx}(\tanh x) = \dfrac{d}{dx}\left(\dfrac{\sinh x}{\cosh x}\right) = \dfrac{\cosh x\,\cosh x - \sinh x\,\sinh x}{\cosh^2 x} = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

(c) $\dfrac{d}{dx}(\operatorname{csch} x) = \dfrac{d}{dx}\left(\dfrac{1}{\sinh x}\right) = -\dfrac{\cosh x}{\sinh^2 x} = -\dfrac{1}{\sinh x} \cdot \dfrac{\cosh x}{\sinh x} = -\operatorname{csch} x\,\coth x$

(d) $\dfrac{d}{dx}(\operatorname{sech} x) = \dfrac{d}{dx}\left(\dfrac{1}{\cosh x}\right) = -\dfrac{\sinh x}{\cosh^2 x} = -\dfrac{1}{\cosh x} \cdot \dfrac{\sinh x}{\cosh x} = -\operatorname{sech} x\,\tanh x$

(e) $\dfrac{d}{dx}(\coth x) = \dfrac{d}{dx}\left(\dfrac{\cosh x}{\sinh x}\right) = \dfrac{\sinh x\,\sinh x - \cosh x\,\cosh x}{\sinh^2 x} = \dfrac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\dfrac{1}{\sinh^2 x} = -\operatorname{csch}^2 x$

25. Let $y = \sinh^{-1} x$. Then $\sinh y = x$ and, by Example 1(a), $\cosh^2 y - \sinh^2 y = 1 \;\Rightarrow\;$ [with $\cosh y > 0$]

$\cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$. So by Exercise 9, $e^y = \sinh y + \cosh y = x + \sqrt{1 + x^2} \;\Rightarrow\; y = \ln\left(x + \sqrt{1 + x^2}\right)$.

26. Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0$, so $\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$. So, by Exercise 9,

$e^y = \cosh y + \sinh y = x + \sqrt{x^2 - 1} \;\Rightarrow\; y = \ln\left(x + \sqrt{x^2 - 1}\right)$.

Another method: Write $x = \cosh y = \frac{1}{2}\left(e^y + e^{-y}\right)$ and solve a quadratic, as in Example 3.

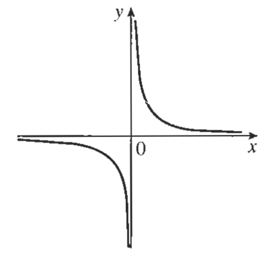
27. (a) Let $y = \tanh^{-1} x$. Then $x = \tanh y = \dfrac{\sinh y}{\cosh y} = \dfrac{(e^y - e^{-y})/2}{(e^y + e^{-y})/2} \cdot \dfrac{e^y}{e^y} = \dfrac{e^{2y} - 1}{e^{2y} + 1} \;\Rightarrow\; xe^{2y} + x = e^{2y} - 1 \;\Rightarrow$

$1 + x = e^{2y} - xe^{2y} \;\Rightarrow\; 1 + x = e^{2y}(1 - x) \;\Rightarrow\; e^{2y} = \dfrac{1+x}{1-x} \;\Rightarrow\; 2y = \ln\left(\dfrac{1+x}{1-x}\right) \;\Rightarrow\; y = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

(b) Let $y = \tanh^{-1} x$. Then $x = \tanh y$, so from Exercise 18 we have

$e^{2y} = \dfrac{1 + \tanh y}{1 - \tanh y} = \dfrac{1+x}{1-x} \;\Rightarrow\; 2y = \ln\left(\dfrac{1+x}{1-x}\right) \;\Rightarrow\; y = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

28. (a) (i) $y = \operatorname{csch}^{-1} x \quad \Leftrightarrow \quad \operatorname{csch} y = x \quad (x \neq 0)$



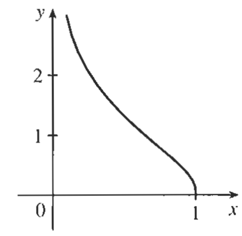
(ii) We sketch the graph of csch^{-1} by reflecting the graph of csch (see Exercise 22) about the line $y = x$.

(iii) Let $y = \operatorname{csch}^{-1} x$. Then $x = \operatorname{csch} y = \dfrac{2}{e^y - e^{-y}} \quad \Rightarrow \quad xe^y - xe^{-y} = 2 \quad \Rightarrow$

$x(e^y)^2 - 2e^y - x = 0 \quad \Rightarrow \quad e^y = \dfrac{1 \pm \sqrt{x^2+1}}{x}$. But $e^y > 0$, so for $x > 0$,

$e^y = \dfrac{1 + \sqrt{x^2+1}}{x}$ and for $x < 0$, $e^y = \dfrac{1 - \sqrt{x^2+1}}{x}$. Thus, $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{x^2+1}}{|x|}\right)$.

(b) (i) $y = \operatorname{sech}^{-1} x \quad \Leftrightarrow \quad \operatorname{sech} y = x$ and $y > 0$.



(ii) We sketch the graph of sech^{-1} by reflecting the graph of sech (see Exercise 22) about the line $y = x$.

(iii) Let $y = \operatorname{sech}^{-1} x$, so $x = \operatorname{sech} y = \dfrac{2}{e^y + e^{-y}} \quad \Rightarrow \quad xe^y + xe^{-y} = 2 \quad \Rightarrow$

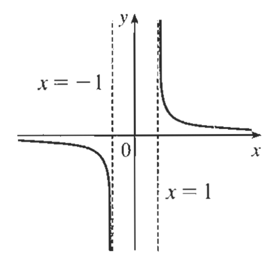
$x\,(e^y)^2 - 2e^y + x = 0 \quad \Leftrightarrow \quad e^y = \dfrac{1 \pm \sqrt{1-x^2}}{x}$. But $y > 0 \quad \Rightarrow \quad e^y > 1$.

This rules out the minus sign because $\dfrac{1 - \sqrt{1-x^2}}{x} > 1 \quad \Leftrightarrow \quad 1 - \sqrt{1-x^2} > x \quad \Leftrightarrow \quad 1 - x > \sqrt{1-x^2} \quad \Leftrightarrow$

$1 - 2x + x^2 > 1 - x^2 \quad \Leftrightarrow \quad x^2 > x \quad \Leftrightarrow \quad x > 1$, but $x = \operatorname{sech} y \leq 1$.

Thus, $e^y = \dfrac{1 + \sqrt{1-x^2}}{x} \quad \Rightarrow \quad \operatorname{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1-x^2}}{x}\right)$.

(c) (i) $y = \coth^{-1} x \quad \Leftrightarrow \quad \coth y = x$



(ii) We sketch the graph of $\coth^{-1}$ by reflecting the graph of coth (see Exercise 22) about the line $y = x$.

(iii) Let $y = \coth^{-1} x$. Then $x = \coth y = \dfrac{e^y + e^{-y}}{e^y - e^{-y}} \quad \Rightarrow$

$xe^y - xe^{-y} = e^y + e^{-y} \quad \Rightarrow \quad (x-1)e^y = (x+1)e^{-y} \quad \Rightarrow \quad e^{2y} = \dfrac{x+1}{x-1} \quad \Rightarrow$

$2y = \ln \dfrac{x+1}{x-1} \quad \Rightarrow \quad \coth^{-1} x = \dfrac{1}{2} \ln \dfrac{x+1}{x-1}$

29. (a) Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0 \quad \Rightarrow \quad \sinh y \dfrac{dy}{dx} = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$ [since $\sinh y \geq 0$ for $y \geq 0$]. *Or:* Use Formula 4.

(b) Let $y = \tanh^{-1} x$. Then $\tanh y = x \quad \Rightarrow \quad \operatorname{sech}^2 y \dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y} = \dfrac{1}{1 - x^2}$.

Or: Use Formula 5.

(c) Let $y = \operatorname{csch}^{-1} x$. Then $\operatorname{csch} y = x \quad \Rightarrow \quad -\operatorname{csch} y \coth y \dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y}$. By Exercise 13,

$\coth y = \pm\sqrt{\operatorname{csch}^2 y + 1} = \pm\sqrt{x^2+1}$. If $x > 0$, then $\coth y > 0$, so $\coth y = \sqrt{x^2+1}$. If $x < 0$, then $\coth y < 0$,

so $\coth y = -\sqrt{x^2+1}$. In either case we have $\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y} = -\dfrac{1}{|x|\sqrt{x^2+1}}$.

(d) Let $y = \operatorname{sech}^{-1} x$. Then $\operatorname{sech} y = x \Rightarrow -\operatorname{sech} y \tanh y \dfrac{dy}{dx} = 1 \Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{sech} y \tanh y} = -\dfrac{1}{\operatorname{sech} y\sqrt{1-\operatorname{sech}^2 y}} = -\dfrac{1}{x\sqrt{1-x^2}}$. [Note that $y > 0$ and so $\tanh y > 0$.]

(e) Let $y = \coth^{-1} x$. Then $\coth y = x \Rightarrow -\operatorname{csch}^2 y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch}^2 y} = \dfrac{1}{1-\coth^2 y} = \dfrac{1}{1-x^2}$

by Exercise 13.

30. $f(x) = \tanh(1+e^{2x}) \Rightarrow f'(x) = \operatorname{sech}^2(1+e^{2x})\dfrac{d}{dx}(1+e^{2x}) = 2e^{2x}\operatorname{sech}^2(1+e^{2x})$

31. $f(x) = x\sinh x - \cosh x \Rightarrow f'(x) = x(\sinh x)' + \sinh x \cdot 1 - \sinh x = x\cosh x$

32. $g(x) = \cosh(\ln x) \Rightarrow g'(x) = \sinh(\ln x)\cdot(\ln x)' = \frac{1}{x}\sinh(\ln x)$

Or: $g(x) = \cosh(\ln x) = \frac{1}{2}(e^{\ln x} + e^{-\ln x}) = \frac{1}{2}(x + x^{-1}) \Rightarrow g'(x) = \frac{1}{2}(1 - x^{-2}) = \frac{1}{2} - 1/(2x^2)$

33. $h(x) = \ln(\cosh x) \Rightarrow h'(x) = \dfrac{1}{\cosh x}(\cosh x)' = \dfrac{\sinh x}{\cosh x} = \tanh x$

34. $y = x\coth(1+x^2) \Rightarrow y' = x[-\operatorname{csch}^2(1+x^2)\cdot 2x] + \coth(1+x^2)\cdot 1 = -2x^2\operatorname{csch}^2(1+x^2) + \coth(1+x^2)$

35. $y = e^{\cosh 3x} \Rightarrow y' = e^{\cosh 3x}\cdot \sinh 3x \cdot 3 = 3e^{\cosh 3x}\sinh 3x$

36. $f(t) = \operatorname{csch} t\,(1 - \ln \operatorname{csch} t) \Rightarrow$

$$f'(t) = \operatorname{csch} t\left[-\frac{1}{\operatorname{csch} t}(-\operatorname{csch} t\,\coth t)\right] + (1 - \ln\operatorname{csch} t)(-\operatorname{csch} t\,\coth t)$$

$$= \operatorname{csch} t\,\coth t - (1 - \ln\operatorname{csch} t)\operatorname{csch} t\,\coth t = \operatorname{csch} t\,\coth t\,[1 - (1 - \ln\operatorname{csch} t)] = \operatorname{csch} t\,\coth t\,\ln\operatorname{csch} t$$

37. $f(t) = \operatorname{sech}^2(e^t) = [\operatorname{sech}(e^t)]^2 \Rightarrow$

$$f'(t) = 2[\operatorname{sech}(e^t)]\,[\operatorname{sech}(e^t)]' = 2\operatorname{sech}(e^t)\left[-\operatorname{sech}(e^t)\tanh(e^t)\cdot e^t\right] = -2e^t\operatorname{sech}^2(e^t)\tanh(e^t)$$

38. $y = \sinh(\cosh x) \Rightarrow y' = \cosh(\cosh x)\cdot\sinh x$

39. $y = \arctan(\tanh x) \Rightarrow y' = \dfrac{1}{1+(\tanh x)^2}(\tanh x)' = \dfrac{\operatorname{sech}^2 x}{1+\tanh^2 x}$

40. $y = \sqrt[4]{\dfrac{1+\tanh x}{1-\tanh x}} = \left(\dfrac{1+\tanh x}{1-\tanh x}\right)^{1/4} \Rightarrow$

$$y' = \frac{1}{4}\left(\frac{1+\tanh x}{1-\tanh x}\right)^{-3/4}\frac{(1-\tanh x)(\operatorname{sech}^2 x) - (1+\tanh x)(-\operatorname{sech}^2 x)}{(1-\tanh x)^2}$$

$$= \frac{1(1-\tanh x)^{3/4}}{4(1+\tanh x)^{3/4}}\,\frac{\operatorname{sech}^2 x\,[(1-\tanh x)+(1+\tanh x)]}{(1-\tanh x)^2} = \frac{2\operatorname{sech}^2 x\,(1-\tanh x)^{3/4}}{4(1+\tanh x)^{3/4}(1-\tanh x)^2}$$

$$= \frac{\operatorname{sech}^2 x}{2(1+\tanh x)^{3/4}(1-\tanh x)^{5/4}}$$

Or: From Exercise 18, $\dfrac{1+\tanh x}{1-\tanh x} = e^{2x}$, so $y = \sqrt[4]{\dfrac{1+\tanh x}{1-\tanh x}} = \sqrt[4]{e^{2x}} = e^{x/2}$ and $y' = \frac{1}{2}e^{x/2}$.

41. $G(x) = \dfrac{1-\cosh x}{1+\cosh x} \Rightarrow$

$$G'(x) = \frac{(1+\cosh x)(-\sinh x)-(1-\cosh x)(\sinh x)}{(1+\cosh x)^2} = \frac{-\sinh x - \sinh x\cosh x - \sinh x + \sinh x\cosh x}{(1+\cosh x)^2} = \frac{-2\sinh x}{(1+\cosh x)^2}$$

42. $y = x^2\sinh^{-1}(2x) \Rightarrow y' = x^2 \cdot \dfrac{1}{\sqrt{1+(2x)^2}} \cdot 2 + \sinh^{-1}(2x)\cdot 2x = 2x\left[\dfrac{x}{\sqrt{1+4x^2}} + \sinh^{-1}(2x)\right]$

43. $y = \tanh^{-1}\sqrt{x} \Rightarrow y' = \dfrac{1}{1-\left(\sqrt{x}\right)^2}\cdot\dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}\,(1-x)}$

44. $y = x\tanh^{-1}x + \ln\sqrt{1-x^2} = x\tanh^{-1}x + \frac{1}{2}\ln(1-x^2) \Rightarrow$

$$y' = \tanh^{-1}x + \frac{x}{1-x^2} + \frac{1}{2}\left(\frac{1}{1-x^2}\right)(-2x) = \tanh^{-1}x$$

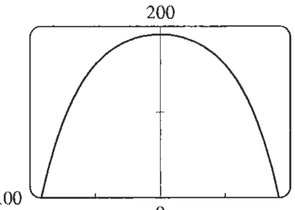
45. $y = x\sinh^{-1}(x/3) - \sqrt{9+x^2} \Rightarrow$

$$y' = \sinh^{-1}\left(\frac{x}{3}\right) + x\frac{1/3}{\sqrt{1+(x/3)^2}} - \frac{2x}{2\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right) + \frac{x}{\sqrt{9+x^2}} - \frac{x}{\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right)$$

46. $y = \operatorname{sech}^{-1}\sqrt{1-x^2} \Rightarrow y' = -\dfrac{1}{\sqrt{1-x^2}\,\sqrt{1-(1-x^2)}}\dfrac{-2x}{2\sqrt{1-x^2}} = \dfrac{x}{(1-x^2)\,|x|} = \dfrac{1}{1-x^2}$ since $x > 0$.

47. $y = \coth^{-1}\sqrt{x^2+1} \Rightarrow y' = \dfrac{1}{1-(x^2+1)}\dfrac{2x}{2\sqrt{x^2+1}} = -\dfrac{1}{x\sqrt{x^2+1}}$

48. (a) Let $a = 0.03291765$. A graph of the central curve, $y = f(x) = 211.49 - 20.96\cosh ax$, is shown.



(b) $f(0) = 211.49 - 20.96\cosh 0 = 211.49 - 20.96(1) = 190.53$ m.

(c) $y = 100 \Rightarrow 100 = 211.49 - 20.96\cosh ax \Rightarrow$

$20.96\cosh ax = 111.49 \Rightarrow \cosh ax = \dfrac{111.49}{20.96} \Rightarrow$

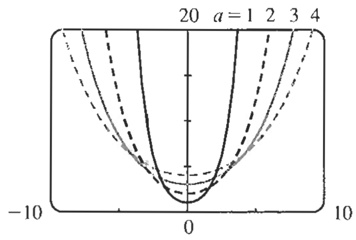
$ax = \pm\cosh^{-1}\dfrac{111.49}{20.96} \Rightarrow x = \pm\dfrac{1}{a}\cosh^{-1}\dfrac{111.49}{20.96} \approx \pm 71.56$ m. The points are approximately $(\pm 71.56, 100)$.

(d) $f(x) = 211.49 - 20.96\cosh ax \Rightarrow f'(x) = -20.96\sinh ax \cdot a$.

$$f'\left(\pm\frac{1}{a}\cosh^{-1}\frac{111.49}{20.96}\right) = -20.96a\sinh\left[a\left(\pm\frac{1}{a}\cosh^{-1}\frac{111.49}{20.96}\right)\right] = -20.96a\sinh\left(\pm\cosh^{-1}\frac{111.49}{20.96}\right) \approx \mp 3.6.$$

So the slope at $(71.56, 100)$ is about -3.6 and the slope at $(-71.56, 100)$ is about 3.6.

49. As the depth d of the water gets large, the fraction $\dfrac{2\pi d}{L}$ gets large, and from Figure 3 or Exercise 23(a), $\tanh\left(\dfrac{2\pi d}{L}\right)$ approaches 1. Thus, $v = \sqrt{\dfrac{gL}{2\pi}\tanh\left(\dfrac{2\pi d}{L}\right)} \approx \sqrt{\dfrac{gL}{2\pi}(1)} = \sqrt{\dfrac{gL}{2\pi}}$.

50. 

For $y = a\cosh(x/a)$ with $a > 0$, we have the y-intercept equal to a. As a increases, the graph flattens.

51. (a) $y = 20\cosh(x/20) - 15 \;\Rightarrow\; y' = 20\sinh(x/20) \cdot \frac{1}{20} = \sinh(x/20)$. Since the right pole is positioned at $x = 7$, we have $y'(7) = \sinh\frac{7}{20} \approx 0.3572$.

(b) If α is the angle between the tangent line and the x-axis, then $\tan\alpha = $ slope of the line $= \sinh\frac{7}{20}$, so $\alpha = \tan^{-1}\left(\sinh\frac{7}{20}\right) \approx 0.343$ rad $\approx 19.66^\circ$. Thus, the angle between the line and the pole is $\theta = 90^\circ - \alpha \approx 70.34^\circ$.

52. We differentiate the function twice, then substitute into the differential equation: $y = \dfrac{T}{\rho g}\cosh\dfrac{\rho g x}{T} \;\Rightarrow$

$\dfrac{dy}{dx} = \dfrac{T}{\rho g}\sinh\left(\dfrac{\rho g x}{T}\right)\dfrac{\rho g}{T} = \sinh\dfrac{\rho g x}{T} \;\Rightarrow\; \dfrac{d^2y}{dx^2} = \cosh\left(\dfrac{\rho g x}{T}\right)\dfrac{\rho g}{T} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$. We evaluate the two sides separately: LHS $= \dfrac{d^2y}{dx^2} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$ and RHS $= \dfrac{\rho g}{T}\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \dfrac{\rho g}{T}\sqrt{1 + \sinh^2\dfrac{\rho g x}{T}} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$, by the identity proved in Example 1(a).

53. (a) $y = A\sinh mx + B\cosh mx \;\Rightarrow\; y' = mA\cosh mx + mB\sinh mx \;\Rightarrow$
$y'' = m^2 A\sinh mx + m^2 B\cosh mx = m^2(A\sinh mx + B\cosh mx) = m^2 y$

(b) From part (a), a solution of $y'' = 9y$ is $y(x) = A\sinh 3x + B\cosh 3x$. So $-4 = y(0) = A\sinh 0 + B\cosh 0 = B$, so $B = -4$. Now $y'(x) = 3A\cosh 3x - 12\sinh 3x \;\Rightarrow\; 6 = y'(0) = 3A \;\Rightarrow\; A = 2$, so $y = 2\sinh 3x - 4\cosh 3x$.

54. $\displaystyle\lim_{x\to\infty}\frac{\sinh x}{e^x} = \lim_{x\to\infty}\frac{e^x - e^{-x}}{2e^x} = \lim_{x\to\infty}\frac{1 - e^{-2x}}{2} = \frac{1-0}{2} = \frac{1}{2}$

55. The tangent to $y = \cosh x$ has slope 1 when $y' = \sinh x = 1 \;\Rightarrow\; x = \sinh^{-1} 1 = \ln\left(1 + \sqrt{2}\right)$, by Equation 3. Since $\sinh x = 1$ and $y = \cosh x = \sqrt{1 + \sinh^2 x}$, we have $\cosh x = \sqrt{2}$. The point is $\left(\ln\left(1+\sqrt{2}\right), \sqrt{2}\right)$.

56. $\cosh x = \cosh[\ln(\sec\theta + \tan\theta)] = \dfrac{1}{2}\left[e^{\ln(\sec\theta + \tan\theta)} + e^{-\ln(\sec\theta+\tan\theta)}\right] = \dfrac{1}{2}\left[\sec\theta + \tan\theta + \dfrac{1}{\sec\theta + \tan\theta}\right]$

$$= \frac{1}{2}\left[\sec\theta + \tan\theta + \frac{\sec\theta - \tan\theta}{(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}\right] = \frac{1}{2}\left[\sec\theta + \tan\theta + \frac{\sec\theta - \tan\theta}{\sec^2\theta - \tan^2\theta}\right]$$

$$= \tfrac{1}{2}(\sec\theta + \tan\theta + \sec\theta - \tan\theta) = \sec\theta$$

57. If $ae^x + be^{-x} = \alpha\cosh(x + \beta)$ [or $\alpha\sinh(x+\beta)$], then
$ae^x + be^{-x} = \frac{\alpha}{2}\left(e^{x+\beta} \pm e^{-x-\beta}\right) = \frac{\alpha}{2}\left(e^x e^\beta \pm e^{-x}e^{-\beta}\right) = \left(\frac{\alpha}{2}e^\beta\right)e^x \pm \left(\frac{\alpha}{2}e^{-\beta}\right)e^{-x}$. Comparing coefficients of e^x

and e^{-x}, we have $a = \frac{\alpha}{2}e^{\beta}$ (1) and $b = \pm\frac{\alpha}{2}e^{-\beta}$ (2). We need to find α and β. Dividing equation (1) by equation (2) gives us $\frac{a}{b} = \pm e^{2\beta}$ $\Rightarrow$ $(\star)$ $2\beta = \ln\left(\pm\frac{a}{b}\right)$ $\Rightarrow$ $\beta = \frac{1}{2}\ln\left(\pm\frac{a}{b}\right)$. Solving equations (1) and (2) for e^{β} gives us

$$e^{\beta} = \frac{2a}{\alpha} \text{ and } e^{\beta} = \pm\frac{\alpha}{2b}, \text{ so } \frac{2a}{\alpha} = \pm\frac{\alpha}{2b} \quad \Rightarrow \quad \alpha^2 = \pm 4ab \quad \Rightarrow \quad \alpha = 2\sqrt{\pm ab}.$$

$(\star)$ If $\frac{a}{b} > 0$, we use the $+$ sign and obtain a cosh function, whereas if $\frac{a}{b} < 0$, we use the $-$ sign and obtain a sinh function.

In summary, if a and b have the same sign, we have $ae^{x} + be^{-x} = 2\sqrt{ab}\cosh\left(x + \frac{1}{2}\ln\frac{a}{b}\right)$, whereas, if a and b have the opposite sign, then $ae^{x} + be^{-x} = 2\sqrt{-ab}\sinh\left(x + \frac{1}{2}\ln\left(-\frac{a}{b}\right)\right)$.