3 Review

CONCEPT CHECK

1. (a) The Power Rule: If n is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$. The derivative of a variable base raised to a constant power is the power times the base raised to the power minus one.

   (b) The Constant Multiple Rule: If c is a constant and f is a differentiable function, then $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$. The derivative of a constant times a function is the constant times the derivative of the function.

   (c) The Sum Rule: If f and g are both differentiable, then $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$. The derivative of a sum of functions is the sum of the derivatives.

   (d) The Difference Rule: If f and g are both differentiable, then $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$. The derivative of a difference of functions is the difference of the derivatives.

   (e) The Product Rule: If f and g are both differentiable, then $\frac{d}{dx}[f(x)\,g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$. The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

   (f) The Quotient Rule: If f and g are both differentiable, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$. The derivative of a quotient of functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

   (g) The Chain Rule: If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product $F'(x) = f'(g(x))\,g'(x)$. The derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

2. (a) $y = x^n \Rightarrow y' = nx^{n-1}$ (b) $y = e^x \Rightarrow y' = e^x$

(c) $y = a^x \Rightarrow y' = a^x \ln a$ (d) $y = \ln x \Rightarrow y' = 1/x$

(e) $y = \log_a x \Rightarrow y' = 1/(x \ln a)$ (f) $y = \sin x \Rightarrow y' = \cos x$

(g) $y = \cos x \Rightarrow y' = -\sin x$ (h) $y = \tan x \Rightarrow y' = \sec^2 x$

(i) $y = \csc x \Rightarrow y' = -\csc x \cot x$ (j) $y = \sec x \Rightarrow y' = \sec x \tan x$

(k) $y = \cot x \Rightarrow y' = -\csc^2 x$ (l) $y = \sin^{-1} x \Rightarrow y' = 1/\sqrt{1-x^2}$

(m) $y = \cos^{-1} x \Rightarrow y' = -1/\sqrt{1-x^2}$ (n) $y = \tan^{-1} x \Rightarrow y' = 1/(1+x^2)$

(o) $y = \sinh x \Rightarrow y' = \cosh x$ (p) $y = \cosh x \Rightarrow y' = \sinh x$

(q) $y = \tanh x \Rightarrow y' = \operatorname{sech}^2 x$ (r) $y = \sinh^{-1} x \Rightarrow y' = 1/\sqrt{1+x^2}$

(s) $y = \cosh^{-1} x \Rightarrow y' = 1/\sqrt{x^2-1}$ (t) $y = \tanh^{-1} x \Rightarrow y' = 1/(1-x^2)$

3. (a) e is the number such that $\lim_{h\to 0} \dfrac{e^h - 1}{h} = 1$.

(b) $e = \lim_{x\to 0} (1+x)^{1/x}$

(c) The differentiation formula for $y = a^x$ $[y' = a^x \ln a]$ is simplest when $a = e$ because $\ln e = 1$.

(d) The differentiation formula for $y = \log_a x$ $[y' = 1/(x \ln a)]$ is simplest when $a = e$ because $\ln e = 1$.

4. (a) Implicit differentiation consists of differentiating both sides of an equation involving x and y with respect to x, and then solving the resulting equation for y'.

(b) Logarithmic differentiation consists of taking natural logarithms of both sides of an equation $y = f(x)$, simplifying, differentiating implicitly with respect to x, and then solving the resulting equation for y'.

5. (a) The linearization L of f at $x = a$ is $L(x) = f(a) + f'(a)(x-a)$.

(b) If $y = f(x)$, then the differential dy is given by $dy = f'(x)\,dx$.

(c) See Figure 5 in Section 3.10.

TRUE-FALSE QUIZ

1. True. This is the Sum Rule.

2. False. See the warning before the Product Rule.

3. True. This is the Chain Rule.

4. True by the Chain Rule.

5. False. $\dfrac{d}{dx} f\left(\sqrt{x}\right) = \dfrac{f'\left(\sqrt{x}\right)}{2\sqrt{x}}$ by the Chain Rule.

6. False. e^2 is a constant, so $y' = 0$.

7. False. $\dfrac{d}{dx}\,10^x = 10^x \ln 10$

8. False. $\ln 10$ is a constant, so its derivative is 0.

9. True. $\dfrac{d}{dx}\,(\tan^2 x) = 2\,\tan x\,\sec^2 x$, and $\dfrac{d}{dx}\,(\sec^2 x) = 2\,\sec x\,(\sec x\,\tan x) = 2\,\tan x\,\sec^2 x$.

Or: $\dfrac{d}{dx}\,(\sec^2 x) = \dfrac{d}{dx}\,(1 + \tan^2 x) = \dfrac{d}{dx}\,(\tan^2 x)$.

10. False. $f(x) = |x^2 + x| = x^2 + x$ for $x \geq 0$ or $x \leq -1$ and $|x^2 + x| = -(x^2 + x)$ for $-1 < x < 0$.
So $f'(x) = 2x + 1$ for $x > 0$ or $x < -1$ and $f'(x) = -(2x + 1)$ for $-1 < x < 0$. But $|2x + 1| = 2x + 1$ for $x \geq -\frac{1}{2}$ and $|2x + 1| = -2x - 1$ for $x < -\frac{1}{2}$.

11. True. $g(x) = x^5 \;\Rightarrow\; g'(x) = 5x^4 \;\Rightarrow\; g'(2) = 5(2)^4 = 80$, and by the definition of the derivative,
$\displaystyle\lim_{x \to 2} \frac{g(x) - g(2)}{x - 2} = g'(2) = 80$.

12. False. A tangent line to the parabola $y = x^2$ has slope $dy/dx = 2x$, so at $(-2, 4)$ the slope of the tangent is $2(-2) = -4$ and an equation of the tangent line is $y - 4 = -4(x + 2)$. [The given equation, $y - 4 = 2x(x + 2)$, is not even linear!]

EXERCISES

1. $y = (x^4 - 3x^2 + 5)^3 \;\Rightarrow$

$$y' = 3(x^4 - 3x^2 + 5)^2\,\frac{d}{dx}\,(x^4 - 3x^2 + 5) = 3(x^4 - 3x^2 + 5)^2(4x^3 - 6x) = 6x(x^4 - 3x^2 + 5)^2(2x^2 - 3)$$

2. $y = \cos(\tan x) \;\Rightarrow\; y' = -\sin(\tan x)\,\dfrac{d}{dx}\,(\tan x) = -\sin(\tan x)(\sec^2 x)$

3. $y = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}} = x^{1/2} + x^{-4/3} \;\Rightarrow\; y' = \frac{1}{2}x^{-1/2} - \frac{4}{3}x^{-7/3} = \dfrac{1}{2\sqrt{x}} - \dfrac{4}{3\,\sqrt[3]{x^7}}$

4. $y = \dfrac{3x - 2}{\sqrt{2x + 1}} \;\Rightarrow$

$$y' = \frac{\sqrt{2x + 1}\,(3) - (3x - 2)\frac{1}{2}(2x + 1)^{-1/2}(2)}{\left(\sqrt{2x + 1}\right)^2} \cdot \frac{(2x + 1)^{1/2}}{(2x + 1)^{1/2}} = \frac{3(2x + 1) - (3x - 2)}{(2x + 1)^{3/2}} = \frac{3x + 5}{(2x + 1)^{3/2}}$$

5. $y = 2x\,\sqrt{x^2 + 1} \;\Rightarrow$

$$y' = 2x \cdot \tfrac{1}{2}(x^2 + 1)^{-1/2}(2x) + \sqrt{x^2 + 1}\,(2) = \frac{2x^2}{\sqrt{x^2 + 1}} + 2\,\sqrt{x^2 + 1} = \frac{2x^2 + 2(x^2 + 1)}{\sqrt{x^2 + 1}} = \frac{2(2x^2 + 1)}{\sqrt{x^2 + 1}}$$

6. $y = \dfrac{e^x}{1 + x^2} \;\Rightarrow\; y' = \dfrac{(1 + x^2)e^x - e^x(2x)}{(1 + x^2)^2} = \dfrac{e^x(x^2 - 2x + 1)}{(1 + x^2)^2} = \dfrac{e^x(x - 1)^2}{(1 + x^2)^2}$

7. $y = e^{\sin 2\theta} \;\Rightarrow\; y' = e^{\sin 2\theta}\,\dfrac{d}{d\theta}\,(\sin 2\theta) = e^{\sin 2\theta}(\cos 2\theta)(2) = 2\cos 2\theta\, e^{\sin 2\theta}$

8. $y = e^{-t}(t^2 - 2t + 2) \;\Rightarrow\; y' = e^{-t}(2t - 2) + (t^2 - 2t + 2)(-e^{-t}) = e^{-t}(2t - 2 - t^2 + 2t - 2) = e^{-t}(-t^2 + 4t - 4)$

9. $y = \dfrac{t}{1-t^2} \;\Rightarrow\; y' = \dfrac{(1-t^2)(1) - t(-2t)}{(1-t^2)^2} = \dfrac{1-t^2+2t^2}{(1-t^2)^2} = \dfrac{t^2+1}{(1-t^2)^2}$

10. $y = e^{mx}\cos nx \;\Rightarrow$

$y' = e^{mx}(\cos nx)' + \cos nx\,(e^{mx})' = e^{mx}(-\sin nx \cdot n) + \cos nx\,(e^{mx} \cdot m) = e^{mx}(m\cos nx - n\sin nx)$

11. $y = \sqrt{x}\cos\sqrt{x} \;\Rightarrow$

$$y' = \sqrt{x}\left(\cos\sqrt{x}\right)' + \cos\sqrt{x}\left(\sqrt{x}\right)' = \sqrt{x}\left[-\sin\sqrt{x}\left(\tfrac{1}{2}x^{-1/2}\right)\right] + \cos\sqrt{x}\left(\tfrac{1}{2}x^{-1/2}\right)$$

$$= \tfrac{1}{2}x^{-1/2}\left(-\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right) = \frac{\cos\sqrt{x} - \sqrt{x}\sin\sqrt{x}}{2\sqrt{x}}$$

12. $y = (\arcsin 2x)^2 \;\Rightarrow\; y' = 2(\arcsin 2x)\cdot(\arcsin 2x)' = 2\arcsin 2x \cdot \dfrac{1}{\sqrt{1-(2x)^2}} \cdot 2 = \dfrac{4\arcsin 2x}{\sqrt{1-4x^2}}$

13. $y = \dfrac{e^{1/x}}{x^2} \;\Rightarrow\; y' = \dfrac{x^2(e^{1/x})' - e^{1/x}(x^2)'}{(x^2)^2} = \dfrac{x^2(e^{1/x})(-1/x^2) - e^{1/x}(2x)}{x^4} = \dfrac{-e^{1/x}(1+2x)}{x^4}$

14. Using the Reciprocal Rule, $g(x) = \dfrac{1}{f(x)} \;\Rightarrow\; g'(x) = -\dfrac{f'(x)}{[f(x)]^2}$, we have $y = \dfrac{1}{\sin(x - \sin x)} \;\Rightarrow$

$y' = -\dfrac{\cos(x-\sin x)(1-\cos x)}{\sin^2(x-\sin x)}$.

15. $\dfrac{d}{dx}(xy^4 + x^2y) = \dfrac{d}{dx}(x+3y) \;\Rightarrow\; x\cdot 4y^3y' + y^4\cdot 1 + x^2\cdot y' + y\cdot 2x = 1 + 3y' \;\Rightarrow$

$y'(4xy^3 + x^2 - 3) = 1 - y^4 - 2xy \;\Rightarrow\; y' = \dfrac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$

16. $y = \ln(\csc 5x) \;\Rightarrow\; y' = \dfrac{1}{\csc 5x}(-\csc 5x\,\cot 5x)(5) = -5\cot 5x$

17. $y = \dfrac{\sec 2\theta}{1+\tan 2\theta} \;\Rightarrow$

$$y' = \frac{(1+\tan 2\theta)(\sec 2\theta\,\tan 2\theta\cdot 2) - (\sec 2\theta)(\sec^2 2\theta\cdot 2)}{(1+\tan 2\theta)^2} = \frac{2\sec 2\theta\,[(1+\tan 2\theta)\tan 2\theta - \sec^2 2\theta]}{(1+\tan 2\theta)^2}$$

$$= \frac{2\sec 2\theta\,(\tan 2\theta + \tan^2 2\theta - \sec^2 2\theta)}{(1+\tan 2\theta)^2} = \frac{2\sec 2\theta\,(\tan 2\theta - 1)}{(1+\tan 2\theta)^2} \qquad [1+\tan^2 x = \sec^2 x]$$

18. $\dfrac{d}{dx}(x^2\cos y + \sin 2y) = \dfrac{d}{dx}(xy) \;\Rightarrow\; x^2(-\sin y\cdot y') + (\cos y)(2x) + \cos 2y\cdot 2y' = x\cdot y' + y\cdot 1 \;\Rightarrow$

$y'(-x^2\sin y + 2\cos 2y - x) = y - 2x\cos y \;\Rightarrow\; y' = \dfrac{y - 2x\cos y}{2\cos 2y - x^2\sin y - x}$

19. $y = e^{cx}(c\sin x - \cos x) \;\Rightarrow$

$y' = e^{cx}(c\cos x + \sin x) + ce^{cx}(c\sin x - \cos x) = e^{cx}(c^2\sin x - c\cos x + c\cos x + \sin x)$

$= e^{cx}(c^2\sin x + \sin x) = e^{cx}\sin x\,(c^2+1)$

20. $y = \ln(x^2e^x) = \ln x^2 + \ln e^x = 2\ln x + x \;\Rightarrow\; y' = 2/x + 1$

21. $y = 3^{x \ln x} \;\Rightarrow\; y' = 3^{x \ln x} \cdot \ln 3 \cdot \dfrac{d}{dx}(x \ln x) = 3^{x \ln x} \cdot \ln 3 \left(x \cdot \dfrac{1}{x} + \ln x \cdot 1 \right) = 3^{x \ln x} \cdot \ln 3 (1 + \ln x)$

22. $y = \sec(1 + x^2) \;\Rightarrow\; y' = 2x \sec(1 + x^2) \tan(1 + x^2)$

23. $y = (1 - x^{-1})^{-1} \;\Rightarrow$

$y' = -1(1 - x^{-1})^{-2}[-(-1x^{-2})] = -(1 - 1/x)^{-2}x^{-2} = -((x-1)/x)^{-2}x^{-2} = -(x-1)^{-2}$

24. $y = (x + \sqrt{x}\,)^{-1/3} \;\Rightarrow\; y' = -\frac{1}{3}(x + \sqrt{x}\,)^{-4/3}\left(1 + \dfrac{1}{2\sqrt{x}}\right)$

25. $\sin(xy) = x^2 - y \;\Rightarrow\; \cos(xy)(xy' + y \cdot 1) = 2x - y' \;\Rightarrow\; x\cos(xy)y' + y' = 2x - y\cos(xy) \;\Rightarrow$

$y'[x\cos(xy) + 1] = 2x - y\cos(xy) \;\Rightarrow\; y' = \dfrac{2x - y\cos(xy)}{x\cos(xy) + 1}$

26. $y = \sqrt{\sin\sqrt{x}} \;\Rightarrow\; y' = \frac{1}{2}\left(\sin\sqrt{x}\right)^{-1/2}\left(\cos\sqrt{x}\right)\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$

27. $y = \log_5(1 + 2x) \;\Rightarrow\; y' = \dfrac{1}{(1 + 2x)\ln 5}\dfrac{d}{dx}(1 + 2x) = \dfrac{2}{(1 + 2x)\ln 5}$

28. $y = (\cos x)^x \;\Rightarrow\; \ln y = \ln(\cos x)^x = x \ln \cos x \;\Rightarrow\; \dfrac{y'}{y} = x \cdot \dfrac{1}{\cos x} \cdot (-\sin x) + \ln \cos x \cdot 1 \;\Rightarrow$

$y' = (\cos x)^x(\ln \cos x - x \tan x)$

29. $y = \ln \sin x - \frac{1}{2}\sin^2 x \;\Rightarrow\; y' = \dfrac{1}{\sin x} \cdot \cos x - \frac{1}{2} \cdot 2\sin x \cdot \cos x = \cot x - \sin x \cos x$

30. $y = \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} \;\Rightarrow$

$$\ln y = \ln \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} = \ln(x^2+1)^4 - \ln[(2x+1)^3(3x-1)^5] = 4\ln(x^2+1) - [\ln(2x+1)^3 + \ln(3x-1)^5]$$

$$= 4\ln(x^2+1) - 3\ln(2x+1) - 5\ln(3x-1) \;\Rightarrow$$

$$\frac{y'}{y} = 4 \cdot \frac{1}{x^2+1} \cdot 2x - 3 \cdot \frac{1}{2x+1} \cdot 2 - 5 \cdot \frac{1}{3x-1} \cdot 3 \;\Rightarrow\; y' = \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\left(\frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1}\right).$$

[The answer could be simplified to $y' = -\dfrac{(x^2 + 56x + 9)(x^2+1)^3}{(2x+1)^4(3x-1)^6}$, but this is unnecessary.]

31. $y = x\tan^{-1}(4x) \;\Rightarrow\; y' = x \cdot \dfrac{1}{1 + (4x)^2} \cdot 4 + \tan^{-1}(4x) \cdot 1 = \dfrac{4x}{1 + 16x^2} + \tan^{-1}(4x)$

32. $y = e^{\cos x} + \cos(e^x) \;\Rightarrow\; y' = e^{\cos x}(-\sin x) + [-\sin(e^x) \cdot e^x] = -\sin x\, e^{\cos x} - e^x \sin(e^x)$

33. $y = \ln|\sec 5x + \tan 5x| \;\Rightarrow$

$y' = \dfrac{1}{\sec 5x + \tan 5x}(\sec 5x \tan 5x \cdot 5 + \sec^2 5x \cdot 5) = \dfrac{5\sec 5x\,(\tan 5x + \sec 5x)}{\sec 5x + \tan 5x} = 5\sec 5x$

34. $y = 10^{\tan \pi\theta} \;\Rightarrow\; y' = 10^{\tan \pi\theta} \cdot \ln 10 \cdot \sec^2 \pi\theta \cdot \pi = \pi(\ln 10)10^{\tan \pi\theta}\sec^2 \pi\theta$

35. $y = \cot(3x^2 + 5) \;\Rightarrow\; y' = -\csc^2(3x^2 + 5)(6x) = -6x\csc^2(3x^2 + 5)$

36. $y = \sqrt{t\ln(t^4)} \quad \Rightarrow$

$$y' = \frac{1}{2}[t\ln(t^4)]^{-1/2}\,\frac{d}{dt}\,[t\ln(t^4)] = \frac{1}{2\sqrt{t\ln(t^4)}} \cdot \left[1 \cdot \ln(t^4) + t \cdot \frac{1}{t^4} \cdot 4t^3\right] = \frac{1}{2\sqrt{t\ln(t^4)}} \cdot [\ln(t^4) + 4] = \frac{\ln(t^4) + 4}{2\sqrt{t\ln(t^4)}}$$

Or: Since y is only defined for $t > 0$, we can write $y = \sqrt{t \cdot 4\ln t} = 2\sqrt{t\ln t}$. Then

$y' = 2 \cdot \dfrac{1}{2\sqrt{t\ln t}} \cdot \left(1 \cdot \ln t + t \cdot \dfrac{1}{t}\right) = \dfrac{\ln t + 1}{\sqrt{t\ln t}}$. This agrees with our first answer since

$$\frac{\ln(t^4) + 4}{2\sqrt{t\ln(t^4)}} = \frac{4\ln t + 4}{2\sqrt{t \cdot 4\ln t}} = \frac{4(\ln t + 1)}{2 \cdot 2\sqrt{t\ln t}} = \frac{\ln t + 1}{\sqrt{t\ln t}}.$$

37. $y = \sin\left(\tan\sqrt{1 + x^3}\right) \quad \Rightarrow \quad y' = \cos\left(\tan\sqrt{1 + x^3}\right)\left(\sec^2\sqrt{1 + x^3}\right)\left[3x^2/\left(2\sqrt{1 + x^3}\right)\right]$

38. $y = \arctan\left(\arcsin\sqrt{x}\right) \quad \Rightarrow \quad y' = \dfrac{1}{1 + \left(\arcsin\sqrt{x}\right)^2} \cdot \dfrac{1}{\sqrt{1 - x}} \cdot \dfrac{1}{2\sqrt{x}}$

39. $y = \tan^2(\sin\theta) = [\tan(\sin\theta)]^2 \quad \Rightarrow \quad y' = 2[\tan(\sin\theta)] \cdot \sec^2(\sin\theta) \cdot \cos\theta$

40. $xe^y = y - 1 \quad \Rightarrow \quad xe^y y' + e^y = y' \quad \Rightarrow \quad e^y = y' - xe^y y' \quad \Rightarrow \quad y' = e^y/(1 - xe^y)$

41. $y = \dfrac{\sqrt{x + 1}\,(2 - x)^5}{(x + 3)^7} \quad \Rightarrow \quad \ln y = \frac{1}{2}\ln(x + 1) + 5\ln(2 - x) - 7\ln(x + 3) \quad \Rightarrow \quad \dfrac{y'}{y} = \dfrac{1}{2(x + 1)} + \dfrac{-5}{2 - x} - \dfrac{7}{x + 3} \quad \Rightarrow$

$$y' = \frac{\sqrt{x + 1}\,(2 - x)^5}{(x + 3)^7}\left[\frac{1}{2(x + 1)} - \frac{5}{2 - x} - \frac{7}{x + 3}\right] \quad \text{or} \quad y' = \frac{(2 - x)^4(3x^2 - 55x - 52)}{2\sqrt{x + 1}\,(x + 3)^8}.$$

42. $y = \dfrac{(x + \lambda)^4}{x^4 + \lambda^4} \quad \Rightarrow \quad y' = \dfrac{(x^4 + \lambda^4)(4)(x + \lambda)^3 - (x + \lambda)^4(4x^3)}{(x^4 + \lambda^4)^2} = \dfrac{4(x + \lambda)^3(\lambda^4 - \lambda x^3)}{(x^4 + \lambda^4)^2}$

43. $y = x\sinh(x^2) \quad \Rightarrow \quad y' = x\cosh(x^2) \cdot 2x + \sinh(x^2) \cdot 1 = 2x^2\cosh(x^2) + \sinh(x^2)$

44. $y = (\sin mx)/x \quad \Rightarrow \quad y' = (mx\cos mx - \sin mx)/x^2$

45. $y = \ln(\cosh 3x) \quad \Rightarrow \quad y' = (1/\cosh 3x)(\sinh 3x)(3) = 3\tanh 3x$

46. $y = \ln\left|\dfrac{x^2 - 4}{2x + 5}\right| = \ln\left|x^2 - 4\right| - \ln|2x + 5| \quad \Rightarrow \quad y' = \dfrac{2x}{x^2 - 4} - \dfrac{2}{2x + 5}$ or $\dfrac{2(x + 1)(x + 4)}{(x + 2)(x - 2)(2x + 5)}$

47. $y = \cosh^{-1}(\sinh x) \quad \Rightarrow \quad y' = \dfrac{1}{\sqrt{(\sinh x)^2 - 1}} \cdot \cosh x = \dfrac{\cosh x}{\sqrt{\sinh^2 x - 1}}$

48. $y = x\tanh^{-1}\sqrt{x} \quad \Rightarrow \quad y' = \tanh^{-1}\sqrt{x} + x\,\dfrac{1}{1 - \left(\sqrt{x}\right)^2}\,\dfrac{1}{2\sqrt{x}} = \tanh^{-1}\sqrt{x} + \dfrac{\sqrt{x}}{2(1 - x)}$

49. $y = \cos\left(e^{\sqrt{\tan 3x}}\right) \quad \Rightarrow$

$$y' = -\sin\left(e^{\sqrt{\tan 3x}}\right) \cdot \left(e^{\sqrt{\tan 3x}}\right)' = -\sin\left(e^{\sqrt{\tan 3x}}\right)\, e^{\sqrt{\tan 3x}} \cdot \tfrac{1}{2}(\tan 3x)^{-1/2} \cdot \sec^2(3x) \cdot 3$$

$$= \frac{-3\sin\left(e^{\sqrt{\tan 3x}}\right)\, e^{\sqrt{\tan 3x}}\,\sec^2(3x)}{2\sqrt{\tan 3x}}$$

50. $y = \sin^2\left(\cos\sqrt{\sin \pi x}\right) = \left[\sin\left(\cos\sqrt{\sin \pi x}\right)\right]^2 \quad\Rightarrow$

$$\begin{aligned}
y' &= 2\left[\sin\left(\cos\sqrt{\sin \pi x}\right)\right]\left[\sin\left(\cos\sqrt{\sin \pi x}\right)\right]' = 2\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\left(\cos\sqrt{\sin \pi x}\right)' \\
&= 2\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\left(-\sin\sqrt{\sin \pi x}\right)\left(\sqrt{\sin \pi x}\right)' \\
&= -2\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\sin\sqrt{\sin \pi x}\cdot\tfrac{1}{2}(\sin \pi x)^{-1/2}(\sin \pi x)' \\
&= \frac{-\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\sin\sqrt{\sin \pi x}}{\sqrt{\sin \pi x}}\cdot\cos \pi x\cdot\pi \\
&= \frac{-\pi\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\sin\sqrt{\sin \pi x}\,\cos \pi x}{\sqrt{\sin \pi x}}
\end{aligned}$$

51. $f(t) = \sqrt{4t+1} \quad\Rightarrow\quad f'(t) = \frac{1}{2}(4t+1)^{-1/2}\cdot 4 = 2(4t+1)^{-1/2} \quad\Rightarrow$

$f''(t) = 2(-\frac{1}{2})(4t+1)^{-3/2}\cdot 4 = -4/(4t+1)^{3/2}$, so $f''(2) = -4/9^{3/2} = -\frac{4}{27}$.

52. $g(\theta) = \theta\sin\theta \quad\Rightarrow\quad g'(\theta) = \theta\cos\theta + \sin\theta\cdot 1 \quad\Rightarrow\quad g''(\theta) = \theta(-\sin\theta) + \cos\theta\cdot 1 + \cos\theta = 2\cos\theta - \theta\sin\theta$,

so $g''(\pi/6) = 2\cos(\pi/6) - (\pi/6)\sin(\pi/6) = 2\left(\sqrt{3}/2\right) - (\pi/6)(1/2) = \sqrt{3} - \pi/12$.

53. $x^6 + y^6 = 1 \quad\Rightarrow\quad 6x^5 + 6y^5y' = 0 \quad\Rightarrow\quad y' = -x^5/y^5 \quad\Rightarrow$

$$y'' = -\frac{y^5(5x^4) - x^5(5y^4y')}{(y^5)^2} = -\frac{5x^4y^4\left[y - x(-x^5/y^5)\right]}{y^{10}} = -\frac{5x^4\left[(y^6+x^6)/y^5\right]}{y^6} = -\frac{5x^4}{y^{11}}$$

54. $f(x) = (2-x)^{-1} \quad\Rightarrow\quad f'(x) = (2-x)^{-2} \quad\Rightarrow\quad f''(x) = 2(2-x)^{-3} \quad\Rightarrow\quad f'''(x) = 2\cdot 3(2-x)^{-4} \quad\Rightarrow$

$f^{(4)}(x) = 2\cdot 3\cdot 4(2-x)^{-5}$. In general, $f^{(n)}(x) = 2\cdot 3\cdot 4\cdot\cdots\cdot n(2-x)^{-(n+1)} = \dfrac{n!}{(2-x)^{(n+1)}}$.

55. We first show it is true for $n = 1$: $f(x) = xe^x \quad\Rightarrow\quad f'(x) = xe^x + e^x = (x+1)e^x$. We now assume it is true for $n = k$: $f^{(k)}(x) = (x+k)e^x$. With this assumption, we must show it is true for $n = k+1$:

$$f^{(k+1)}(x) = \frac{d}{dx}\left[f^{(k)}(x)\right] = \frac{d}{dx}\left[(x+k)e^x\right] = (x+k)e^x + e^x = [(x+k)+1]e^x = [x+(k+1)]e^x.$$

Therefore, $f^{(n)}(x) = (x+n)e^x$ by mathematical induction.

56. $$\lim_{t\to 0}\frac{t^3}{\tan^3 2t} = \lim_{t\to 0}\frac{t^3\cos^3 2t}{\sin^3 2t} = \lim_{t\to 0}\cos^3 2t\cdot\frac{1}{8\dfrac{\sin^3 2t}{(2t)^3}} = \lim_{t\to 0}\frac{\cos^3 2t}{8\left(\lim\limits_{t\to 0}\dfrac{\sin 2t}{2t}\right)^3} = \frac{1}{8\cdot 1^3} = \frac{1}{8}$$

57. $y = 4\sin^2 x \quad\Rightarrow\quad y' = 4\cdot 2\sin x\cos x$. At $\left(\frac{\pi}{6}, 1\right)$, $y' = 8\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$, so an equation of the tangent line is $y - 1 = 2\sqrt{3}\left(x - \frac{\pi}{6}\right)$, or $y = 2\sqrt{3}\,x + 1 - \pi\sqrt{3}/3$.

58. $y = \dfrac{x^2-1}{x^2+1} \quad\Rightarrow\quad y' = \dfrac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$.

At $(0, -1)$, $y' = 0$, so an equation of the tangent line is $y + 1 = 0(x - 0)$, or $y = -1$.

59. $y = \sqrt{1 + 4\sin x} \;\Rightarrow\; y' = \frac{1}{2}(1 + 4\sin x)^{-1/2} \cdot 4\cos x = \dfrac{2\cos x}{\sqrt{1 + 4\sin x}}$.

At $(0, 1)$, $y' = \dfrac{2}{\sqrt{1}} = 2$, so an equation of the tangent line is $y - 1 = 2(x - 0)$, or $y = 2x + 1$.

60. $x^2 + 4xy + y^2 = 13 \;\Rightarrow\; 2x + 4(xy' + y \cdot 1) + 2yy' = 0 \;\Rightarrow\; x + 2xy' + 2y + yy' = 0 \;\Rightarrow$

$2xy' + yy' = -x - 2y \;\Rightarrow\; y'(2x + y) = -x - 2y \;\Rightarrow\; y' = \dfrac{-x - 2y}{2x + y}$.

At $(2, 1)$, $y' = \dfrac{-2 - 2}{4 + 1} = -\dfrac{4}{5}$, so an equation of the tangent line is $y - 1 = -\frac{4}{5}(x - 2)$, or $y = -\frac{4}{5}x + \frac{13}{5}$.

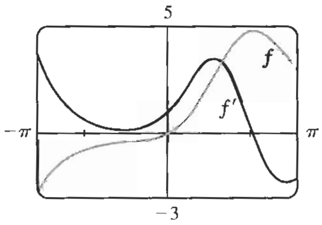
The slope of the normal line is $\frac{5}{4}$, so an equation of the normal line is $y - 1 = \frac{5}{4}(x - 2)$, or $y = \frac{5}{4}x - \frac{3}{2}$.

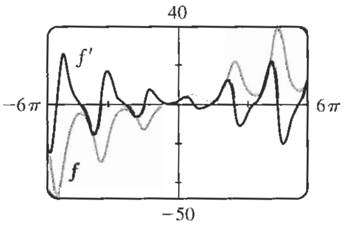
61. $y = (2 + x)e^{-x} \;\Rightarrow\; y' = (2 + x)(-e^{-x}) + e^{-x} \cdot 1 = e^{-x}[-(2 + x) + 1] = e^{-x}(-x - 1)$.

At $(0, 2)$, $y' = 1(-1) = -1$, so an equation of the tangent line is $y - 2 = -1(x - 0)$, or $y = -x + 2$.

The slope of the normal line is 1, so an equation of the normal line is $y - 2 = 1(x - 0)$, or $y = x + 2$.

62. $f(x) = xe^{\sin x} \;\Rightarrow\; f'(x) = x[e^{\sin x}(\cos x)] + e^{\sin x}(1) = e^{\sin x}(x\cos x + 1)$. As a check on our work, we notice from the graphs that $f'(x) > 0$ when f is increasing. Also, we see in the larger viewing rectangle a certain similarity in the graphs of f and f': the sizes of the oscillations of f and f' are linked.





63. (a) $f(x) = x\sqrt{5 - x} \;\Rightarrow$

$$f'(x) = x\left[\frac{1}{2}(5 - x)^{-1/2}(-1)\right] + \sqrt{5 - x} = \frac{-x}{2\sqrt{5 - x}} + \sqrt{5 - x} \cdot \frac{2\sqrt{5 - x}}{2\sqrt{5 - x}} = \frac{-x}{2\sqrt{5 - x}} + \frac{2(5 - x)}{2\sqrt{5 - x}}$$

$$= \frac{-x + 10 - 2x}{2\sqrt{5 - x}} = \frac{10 - 3x}{2\sqrt{5 - x}}$$

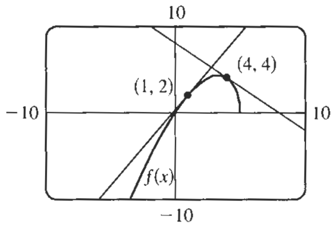
(b) At $(1, 2)$: $f'(1) = \frac{7}{4}$.

So an equation of the tangent line is $y - 2 = \frac{7}{4}(x - 1)$ or $y = \frac{7}{4}x + \frac{1}{4}$.

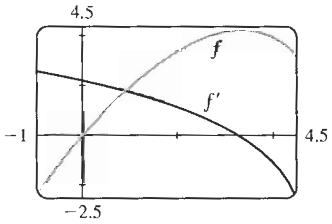
At $(4, 4)$: $f'(4) = -\frac{2}{2} = -1$.

So an equation of the tangent line is $y - 4 = -1(x - 4)$ or $y = -x + 8$.

(c)



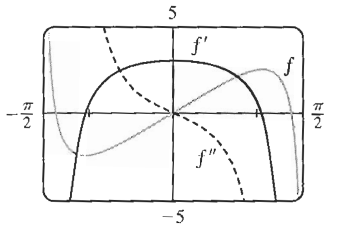
(d)



The graphs look reasonable, since f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

64. (a) $f(x) = 4x - \tan x \quad \Rightarrow \quad f'(x) = 4 - \sec^2 x \quad \Rightarrow \quad f''(x) = -2\sec x\,(\sec x\,\tan x) = -2\sec^2 x\,\tan x$.

(b)



We can see that our answers are reasonable, since the graph of f' is 0 where f has a horizontal tangent, and the graph of f' is positive where f has tangents with positive slope and negative where f has tangents with negative slope. The same correspondence holds between the graphs of f' and f''.

65. $y = \sin x + \cos x \quad \Rightarrow \quad y' = \cos x - \sin x = 0 \quad \Leftrightarrow \quad \cos x = \sin x$ and $0 \le x \le 2\pi \quad \Leftrightarrow \quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$, so the points are $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$.

66. $x^2 + 2y^2 = 1 \quad \Rightarrow \quad 2x + 4yy' = 0 \quad \Rightarrow \quad y' = -x/(2y) = 1 \quad \Leftrightarrow \quad x = -2y$. Since the points lie on the ellipse, we have $(-2y)^2 + 2y^2 = 1 \quad \Rightarrow \quad 6y^2 = 1 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{6}}$. The points are $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.

67. $f(x) = (x-a)(x-b)(x-c) \quad \Rightarrow \quad f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$.

So $\dfrac{f'(x)}{f(x)} = \dfrac{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}{(x-a)(x-b)(x-c)} = \dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c}$.

Or: $f(x) = (x-a)(x-b)(x-c) \quad \Rightarrow \quad \ln|f(x)| = \ln|x-a| + \ln|x-b| + \ln|x-c| \quad \Rightarrow$

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}$$

68. (a) $\cos 2x = \cos^2 x - \sin^2 x \quad \Rightarrow \quad -2\sin 2x = -2\cos x\sin x - 2\sin x\cos x \quad \Leftrightarrow \quad \sin 2x = 2\sin x\cos x$

(b) $\sin(x+a) = \sin x\,\cos a + \cos x\,\sin a \quad \Rightarrow \quad \cos(x+a) = \cos x\,\cos a - \sin x\,\sin a$.

69. (a) $h(x) = f(x)\,g(x) \quad \Rightarrow \quad h'(x) = f(x)\,g'(x) + g(x)\,f'(x) \quad \Rightarrow$

$h'(2) = f(2)\,g'(2) + g(2)\,f'(2) = (3)(4) + (5)(-2) = 12 - 10 = 2$

(b) $F(x) = f(g(x)) \quad \Rightarrow \quad F'(x) = f'(g(x))\,g'(x) \quad \Rightarrow \quad F'(2) = f'(g(2))\,g'(2) = f'(5)(4) = 11 \cdot 4 = 44$

70. (a) $P(x) = f(x)\,g(x) \quad \Rightarrow \quad P'(x) = f(x)\,g'(x) + g(x)\,f'(x) \quad \Rightarrow$

$P'(2) = f(2)\,g'(2) + g(2)\,f'(2) = (1)\left(\frac{6-0}{3-0}\right) + (4)\left(\frac{0-3}{3-0}\right) = (1)(2) + (4)(-1) = 2 - 4 = -2$

(b) $Q(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad Q'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2} \quad \Rightarrow$

$Q'(2) = \dfrac{g(2)\,f'(2) - f(2)\,g'(2)}{[g(2)]^2} = \dfrac{(4)(-1) - (1)(2)}{4^2} = \dfrac{-6}{16} = -\dfrac{3}{8}$

(c) $C(x) = f(g(x)) \quad \Rightarrow \quad C'(x) = f'(g(x))g'(x) \quad \Rightarrow$

$C'(2) = f'(g(2))g'(2) = f'(4)g'(2) = \left(\frac{6-0}{5-3}\right)(2) = (3)(2) = 6$

71. $f(x) = x^2 g(x) \quad \Rightarrow \quad f'(x) = x^2 g'(x) + g(x)(2x) = x[xg'(x) + 2g(x)]$

72. $f(x) = g(x^2) \quad \Rightarrow \quad f'(x) = g'(x^2)(2x) = 2xg'(x^2)$

73. $f(x) = [g(x)]^2 \quad \Rightarrow \quad f'(x) = 2[g(x)] \cdot g'(x) = 2g(x)\,g'(x)$

74. $f(x) = g(g(x)) \quad \Rightarrow \quad f'(x) = g'(g(x))\,g'(x)$

75. $f(x) = g(e^x) \quad \Rightarrow \quad f'(x) = g'(e^x)\,e^x$

76. $f(x) = e^{g(x)} \quad \Rightarrow \quad f'(x) = e^{g(x)}g'(x)$

77. $f(x) = \ln|g(x)| \quad \Rightarrow \quad f'(x) = \dfrac{1}{g(x)}g'(x) = \dfrac{g'(x)}{g(x)}$

78. $f(x) = g(\ln x) \quad \Rightarrow \quad f'(x) = g'(\ln x)\cdot\dfrac{1}{x} = \dfrac{g'(\ln x)}{x}$

79. $h(x) = \dfrac{f(x)\,g(x)}{f(x)+g(x)} \quad \Rightarrow$

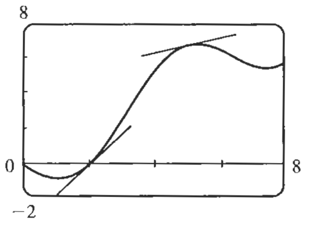
$$\begin{aligned} h'(x) &= \frac{[f(x)+g(x)]\,[f(x)\,g'(x)+g(x)\,f'(x)] - f(x)\,g(x)\,[f'(x)+g'(x)]}{[f(x)+g(x)]^2} \\ &= \frac{[f(x)]^2\,g'(x) + f(x)\,g(x)\,f'(x) + f(x)\,g(x)\,g'(x) + [g(x)]^2\,f'(x) - f(x)\,g(x)\,f'(x) - f(x)\,g(x)\,g'(x)}{[f(x)+g(x)]^2} \\ &= \frac{f'(x)\,[g(x)]^2 + g'(x)\,[f(x)]^2}{[f(x)+g(x)]^2} \end{aligned}$$

80. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}} \quad \Rightarrow \quad h'(x) = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{2\sqrt{f(x)/g(x)}\,[g(x)]^2} = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{2[g(x)]^{3/2}\sqrt{f(x)}}$

81. Using the Chain Rule repeatedly, $h(x) = f(g(\sin 4x)) \quad \Rightarrow$

$h'(x) = f'(g(\sin 4x))\cdot\dfrac{d}{dx}\,(g(\sin 4x)) = f'(g(\sin 4x))\cdot g'(\sin 4x)\cdot\dfrac{d}{dx}\,(\sin 4x) = f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4).$

82. (a)



(b) The average rate of change is larger on $[2, 3]$.

(c) The instantaneous rate of change (the slope of the tangent) is larger at $x = 2$.

(d) $f(x) = x - 2\sin x \quad \Rightarrow \quad f'(x) = 1 - 2\cos x$,

so $f'(2) = 1 - 2\cos 2 \approx 1.8323$ and $f'(5) = 1 - 2\cos 5 \approx 0.4327$.

So $f'(2) > f'(5)$, as predicted in part (c).

83. $y = [\ln(x+4)]^2 \quad \Rightarrow \quad y' = 2[\ln(x+4)]^1\cdot\dfrac{1}{x+4}\cdot 1 = 2\,\dfrac{\ln(x+4)}{x+4}$ and $y' = 0 \quad \Leftrightarrow \quad \ln(x+4) = 0 \quad \Leftrightarrow$

$x + 4 = e^0 \quad \Rightarrow \quad x + 4 = 1 \quad \Leftrightarrow \quad x = -3$, so the tangent is horizontal at the point $(-3, 0)$.

84. (a) The line $x - 4y = 1$ has slope $\frac{1}{4}$. A tangent to $y = e^x$ has slope $\frac{1}{4}$ when $y' = e^x = \frac{1}{4} \quad \Rightarrow \quad x = \ln\frac{1}{4} = -\ln 4$.

Since $y = e^x$, the y-coordinate is $\frac{1}{4}$ and the point of tangency is $\left(-\ln 4, \frac{1}{4}\right)$. Thus, an equation of the tangent line is $y - \frac{1}{4} = \frac{1}{4}(x + \ln 4)$ or $y = \frac{1}{4}x + \frac{1}{4}(\ln 4 + 1)$.

(b) The slope of the tangent at the point (a, e^a) is $\left.\dfrac{d}{dx}\,e^x\right|_{x=a} = e^a$. Thus, an equation of the tangent line is

$y - e^a = e^a(x - a)$. We substitute $x = 0$, $y = 0$ into this equation, since we want the line to pass through the origin:

$0 - e^a = e^a(0 - a) \quad \Leftrightarrow \quad -e^a = e^a(-a) \quad \Leftrightarrow \quad a = 1$. So an equation of the tangent line at the point $(a, e^a) = (1, e)$ is $y - e = e(x - 1)$ or $y = ex$.

85. $y = f(x) = ax^2 + bx + c \Rightarrow f'(x) = 2ax + b$. We know that $f'(-1) = 6$ and $f'(5) = -2$, so $-2a + b = 6$ and $10a + b = -2$. Subtracting the first equation from the second gives $12a = -8 \Rightarrow a = -\frac{2}{3}$. Substituting $-\frac{2}{3}$ for a in the first equation gives $b = \frac{14}{3}$. Now $f(1) = 4 \Rightarrow 4 = a + b + c$, so $c = 4 + \frac{2}{3} - \frac{14}{3} = 0$ and hence, $f(x) = -\frac{2}{3}x^2 + \frac{14}{3}x$.

86. (a) $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty} [K(e^{-at} - e^{-bt})] = K \lim\limits_{t\to\infty} (e^{-at} - e^{-bt}) = K(0 - 0) = 0$ because $-at \to -\infty$ and $-bt \to -\infty$ as $t \to \infty$.

(b) $C(t) = K(e^{-at} - e^{-bt}) \Rightarrow C'(t) = K(e^{-at}(-a) - e^{-bt}(-b)) = K(-ae^{-at} + be^{-bt})$

(c) $C'(t) = 0 \Leftrightarrow be^{-bt} = ae^{-at} \Leftrightarrow \dfrac{b}{a} = e^{(-a+b)t} \Leftrightarrow \ln\dfrac{b}{a} = (b - a)t \Leftrightarrow t = \dfrac{\ln(b/a)}{b - a}$

87. $s(t) = Ae^{-ct}\cos(\omega t + \delta) \Rightarrow$

$v(t) = s'(t) = A\{e^{-ct}[-\omega\sin(\omega t + \delta)] + \cos(\omega t + \delta)(-ce^{-ct})\} = -Ae^{-ct}[\omega\sin(\omega t + \delta) + c\cos(\omega t + \delta)] \Rightarrow$

$$\begin{aligned} a(t) = v'(t) &= -A\{e^{-ct}[\omega^2\cos(\omega t + \delta) - c\omega\sin(\omega t + \delta)] + [\omega\sin(\omega t + \delta) + c\cos(\omega t + \delta)](-ce^{-ct})\} \\ &= -Ae^{-ct}[\omega^2\cos(\omega t + \delta) - c\omega\sin(\omega t + \delta) - c\omega\sin(\omega t + \delta) - c^2\cos(\omega t + \delta)] \\ &= -Ae^{-ct}[(\omega^2 - c^2)\cos(\omega t + \delta) - 2c\omega\sin(\omega t + \delta)] = Ae^{-ct}[(c^2 - \omega^2)\cos(\omega t + \delta) + 2c\omega\sin(\omega t + \delta)] \end{aligned}$$

88. (a) $x = \sqrt{b^2 + c^2t^2} \Rightarrow v(t) = x' = \left[1/\left(2\sqrt{b^2 + c^2t^2}\right)\right] 2c^2t = c^2t/\sqrt{b^2 + c^2t^2} \Rightarrow$

$$a(t) = v'(t) = \frac{c^2\sqrt{b^2 + c^2t^2} - c^2t\left(c^2t/\sqrt{b^2 + c^2t^2}\right)}{b^2 + c^2t^2} = \frac{b^2c^2}{(b^2 + c^2t^2)^{3/2}}$$

(b) $v(t) > 0$ for $t > 0$, so the particle always moves in the positive direction.

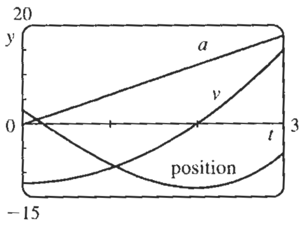
89. (a) $y = t^3 - 12t + 3 \Rightarrow v(t) = y' = 3t^2 - 12 \Rightarrow a(t) = v'(t) = 6t$

(b) $v(t) = 3(t^2 - 4) > 0$ when $t > 2$, so it moves upward when $t > 2$ and downward when $0 \le t < 2$.

(c) Distance upward $= y(3) - y(2) = -6 - (-13) = 7$,

Distance downward $= y(0) - y(2) = 3 - (-13) = 16$. Total distance $= 7 + 16 = 23$.

(d)



(e) The particle is speeding up when v and a have the same sign, that is, when $t > 2$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 2$.

90. (a) $V = \frac{1}{3}\pi r^2 h \Rightarrow dV/dh = \frac{1}{3}\pi r^2$ [r constant]

(b) $V = \frac{1}{3}\pi r^2 h \Rightarrow dV/dr = \frac{2}{3}\pi rh$ [h constant]

91. The linear density ρ is the rate of change of mass m with respect to length x.

$m = x\left(1 + \sqrt{x}\right) = x + x^{3/2} \Rightarrow \rho = dm/dx = 1 + \frac{3}{2}\sqrt{x}$, so the linear density when $x = 4$ is $1 + \frac{3}{2}\sqrt{4} = 4$ kg/m.

92. (a) $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3 \quad \Rightarrow \quad C'(x) = 2 - 0.04x + 0.00021x^2$

(b) $C'(100) = 2 - 4 + 2.1 = \$0.10/\text{unit}$. This value represents the rate at which costs are increasing as the hundredth unit is produced, and is the approximate cost of producing the 101st unit.

(c) The cost of producing the 101st item is $C(101) - C(100) = 990.10107 - 990 = \$0.10107$, slightly larger than $C'(100)$.

93. (a) $y(t) = y(0)e^{kt} = 200e^{kt} \quad \Rightarrow \quad y(0.5) = 200e^{0.5k} = 360 \quad \Rightarrow \quad e^{0.5k} = 1.8 \quad \Rightarrow \quad 0.5k = \ln 1.8 \quad \Rightarrow$
$k = 2\ln 1.8 = \ln(1.8)^2 = \ln 3.24 \quad \Rightarrow \quad y(t) = 200e^{(\ln 3.24)t} = 200(3.24)^t$

(b) $y(4) = 200(3.24)^4 \approx 22{,}040$ bacteria

(c) $y'(t) = 200(3.24)^t \cdot \ln 3.24$, so $y'(4) = 200(3.24)^4 \cdot \ln 3.24 \approx 25{,}910$ bacteria per hour

(d) $200(3.24)^t = 10{,}000 \quad \Rightarrow \quad (3.24)^t = 50 \quad \Rightarrow \quad t\ln 3.24 = \ln 50 \quad \Rightarrow \quad t = \ln 50/\ln 3.24 \approx 3.33$ hours

94. (a) If $y(t)$ is the mass remaining after t years, then $y(t) = y(0)e^{kt} = 100e^{kt}$. $y(5.24) = 100e^{5.24k} = \frac{1}{2} \cdot 100 \quad \Rightarrow$
$e^{5.24k} = \frac{1}{2} \quad \Rightarrow \quad 5.24k = -\ln 2 \quad \Rightarrow \quad k = -\frac{1}{5.24}\ln 2 \quad \Rightarrow \quad y(t) = 100e^{-(\ln 2)t/5.24} = 100 \cdot 2^{-t/5.24}$. Thus,
$y(20) = 100 \cdot 2^{-20/5.24} \approx 7.1$ mg.

(b) $100 \cdot 2^{-t/5.24} = 1 \quad \Rightarrow \quad 2^{-t/5.24} = \dfrac{1}{100} \quad \Rightarrow \quad -\dfrac{t}{5.24}\ln 2 = \ln\dfrac{1}{100} \quad \Rightarrow \quad t = 5.24\,\dfrac{\ln 100}{\ln 2} \approx 34.8$ years

95. (a) $C'(t) = -kC(t) \quad \Rightarrow \quad C(t) = C(0)e^{-kt}$ by Theorem 9.4.2. But $C(0) = C_0$, so $C(t) = C_0e^{-kt}$.

(b) $C(30) = \frac{1}{2}C_0$ since the concentration is reduced by half. Thus, $\frac{1}{2}C_0 = C_0e^{-30k} \quad \Rightarrow \ln\frac{1}{2} = -30k \quad \Rightarrow$
$k = -\frac{1}{30}\ln\frac{1}{2} = \frac{1}{30}\ln 2$. Since 10% of the original concentration remains if 90% is eliminated, we want the value of t such that $C(t) = \frac{1}{10}C_0$. Therefore, $\frac{1}{10}C_0 = C_0e^{-t(\ln 2)/30} \quad \Rightarrow \quad \ln 0.1 = -t(\ln 2)/30 \quad \Rightarrow \quad t = -\frac{30}{\ln 2}\ln 0.1 \approx 100$ h.

96. (a) If $y = u - 20$, $u(0) = 80 \quad \Rightarrow \quad y(0) = 80 - 20 = 60$, and the initial-value problem is $dy/dt = ky$ with $y(0) = 60$.
So the solution is $y(t) = 60e^{kt}$. Now $y(0.5) = 60e^{k(0.5)} = 60 - 20 \quad \Rightarrow \quad e^{0.5k} = \frac{40}{60} = \frac{2}{3} \quad \Rightarrow \quad k = 2\ln\frac{2}{3} = \ln\frac{4}{9}$,
so $y(t) = 60e^{(\ln 4/9)t} = 60(\frac{4}{9})^t$. Thus, $y(1) = 60(\frac{4}{9})^1 = \frac{80}{3} = 26\frac{2}{3}$ °C and $u(1) = 46\frac{2}{3}$ °C.

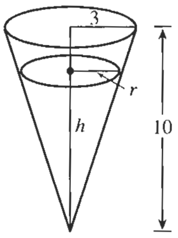
(b) $u(t) = 40 \quad \Rightarrow \quad y(t) = 20.\quad y(t) = 60\left(\dfrac{4}{9}\right)^t = 20 \quad \Rightarrow \quad \left(\dfrac{4}{9}\right)^t = \dfrac{1}{3} \quad \Rightarrow \quad t\ln\dfrac{4}{9} = \ln\dfrac{1}{3} \quad \Rightarrow \quad t = \dfrac{\ln\frac{1}{3}}{\ln\frac{4}{9}} \approx 1.35$ h
or 81.3 min.

97. If x = edge length, then $V = x^3 \quad \Rightarrow \quad dV/dt = 3x^2\,dx/dt = 10 \quad \Rightarrow \quad dx/dt = 10/(3x^2)$ and $S = 6x^2 \quad \Rightarrow$
$dS/dt = (12x)\,dx/dt = 12x[10/(3x^2)] = 40/x$. When $x = 30$, $dS/dt = \frac{40}{30} = \frac{4}{3}\ \text{cm}^2/\text{min}$.

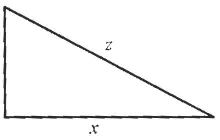
98. Given $dV/dt = 2$, find dh/dt when $h = 5$. $V = \frac{1}{3}\pi r^2 h$ and, from similar triangles, $\dfrac{r}{h} = \dfrac{3}{10} \quad \Rightarrow \quad V = \dfrac{\pi}{3}\left(\dfrac{3h}{10}\right)^2 h = \dfrac{3\pi}{100}h^3$, so

$$2 = \frac{dV}{dt} = \frac{9\pi}{100}h^2\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{200}{9\pi h^2} = \frac{200}{9\pi\,(5)^2} = \frac{8}{9\pi}\ \text{cm/s}$$

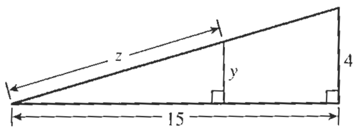
when $h = 5$.



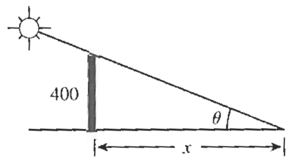
99. Given $dh/dt = 5$ and $dx/dt = 15$, find dz/dt. $z^2 = x^2 + h^2 \;\Rightarrow$
$2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 2h\,\dfrac{dh}{dt} \;\Rightarrow\; \dfrac{dz}{dt} = \dfrac{1}{z}(15x + 5h)$. When $t = 3$,
$h = 45 + 3(5) = 60$ and $x = 15(3) = 45 \;\Rightarrow\; z = \sqrt{45^2 + 60^2} = 75$,
so $\dfrac{dz}{dt} = \dfrac{1}{75}[15(45) + 5(60)] = 13$ ft/s.



100. We are given $dz/dt = 30$ ft/s. By similar triangles, $\dfrac{y}{z} = \dfrac{4}{\sqrt{241}} \;\Rightarrow$
$y = \dfrac{4}{\sqrt{241}}z$, so $\dfrac{dy}{dt} = \dfrac{4}{\sqrt{241}}\dfrac{dz}{dt} = \dfrac{120}{\sqrt{241}} \approx 7.7$ ft/s.



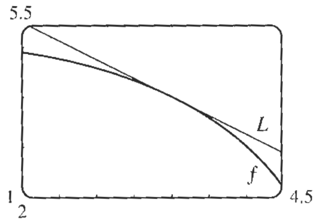
101. We are given $d\theta/dt = -0.25$ rad/h. $\tan\theta = 400/x \;\Rightarrow$
$x = 400\cot\theta \;\Rightarrow\; \dfrac{dx}{dt} = -400\csc^2\theta\,\dfrac{d\theta}{dt}$. When $\theta = \frac{\pi}{6}$,
$\dfrac{dx}{dt} = -400(2)^2(-0.25) = 400$ ft/h.



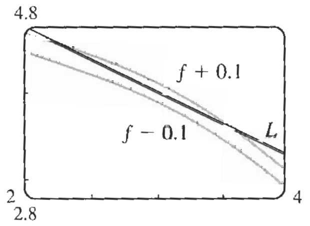
102. (a) $f(x) = \sqrt{25 - x^2} \;\Rightarrow\; f'(x) = \dfrac{-2x}{2\sqrt{25 - x^2}} = -x(25 - x^2)^{-1/2}$.

So the linear approximation to $f(x)$ near 3
is $f(x) \approx f(3) + f'(3)(x - 3) = 4 - \frac{3}{4}(x - 3)$.

(b)

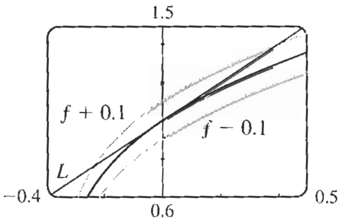


(c) For the required accuracy, we want $\sqrt{25 - x^2} - 0.1 < 4 - \frac{3}{4}(x - 3)$ and
$4 - \frac{3}{4}(x - 3) < \sqrt{25 - x^2} + 0.1$. From the graph, it appears that these both
hold for $2.24 < x < 3.66$.

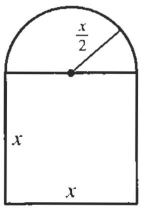


103. (a) $f(x) = \sqrt[3]{1 + 3x} = (1 + 3x)^{1/3} \;\Rightarrow\; f'(x) = (1 + 3x)^{-2/3}$, so the linearization of f at $a = 0$ is
$L(x) = f(0) + f'(0)(x - 0) = 1^{1/3} + 1^{-2/3}x = 1 + x$. Thus, $\sqrt[3]{1 + 3x} \approx 1 + x \;\Rightarrow$
$\sqrt[3]{1.03} = \sqrt[3]{1 + 3(0.01)} \approx 1 + (0.01) = 1.01$.

(b) The linear approximation is $\sqrt[3]{1 + 3x} \approx 1 + x$, so for the required accuracy
we want $\sqrt[3]{1 + 3x} - 0.1 < 1 + x < \sqrt[3]{1 + 3x} + 0.1$. From the graph,
it appears that this is true when $-0.23 < x < 0.40$.



104. $y = x^3 - 2x^2 + 1 \;\Rightarrow\; dy = (3x^2 - 4x)\,dx$. When $x = 2$ and $dx = 0.2$, $dy = \left[3(2)^2 - 4(2)\right](0.2) = 0.8$.



105. $A = x^2 + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = \left(1+\frac{\pi}{8}\right)x^2 \quad\Rightarrow\quad dA = \left(2+\frac{\pi}{4}\right)x\,dx$. When $x = 60$ and $dx = 0.1$, $dA = \left(2+\frac{\pi}{4}\right)60(0.1) = 12 + \frac{3\pi}{2}$, so the maximum error is approximately $12 + \frac{3\pi}{2} \approx 16.7\text{ cm}^2$.

106. $\displaystyle\lim_{x\to 1}\frac{x^{17}-1}{x-1} = \left[\frac{d}{dx}x^{17}\right]_{x\,=\,1} = 17(1)^{16} = 17$

107. $\displaystyle\lim_{h\to 0}\frac{\sqrt[4]{16+h}-2}{h} = \left[\frac{d}{dx}\sqrt[4]{x}\right]_{x\,=\,16} = \frac{1}{4}x^{-3/4}\bigg|_{x\,=\,16} = \frac{1}{4\left(\sqrt[4]{16}\right)^3} = \frac{1}{32}$

108. $\displaystyle\lim_{\theta\to\pi/3}\frac{\cos\theta-0.5}{\theta-\pi/3} = \left[\frac{d}{d\theta}\cos\theta\right]_{\theta\,=\,\pi/3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

109.
$$\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3} = \lim_{x\to 0}\frac{\left(\sqrt{1+\tan x}-\sqrt{1+\sin x}\right)\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}$$
$$= \lim_{x\to 0}\frac{(1+\tan x)-(1+\sin x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)} = \lim_{x\to 0}\frac{\sin x\,(1/\cos x - 1)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\cdot\frac{\cos x}{\cos x}$$
$$= \lim_{x\to 0}\frac{\sin x\,(1-\cos x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x}\cdot\frac{1+\cos x}{1+\cos x}$$
$$= \lim_{x\to 0}\frac{\sin x\cdot\sin^2 x}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}$$
$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^3\lim_{x\to 0}\frac{1}{\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}$$
$$= 1^3\cdot\frac{1}{\left(\sqrt{1}+\sqrt{1}\right)\cdot 1\cdot(1+1)} = \frac{1}{4}$$

110. Differentiating the first given equation implicitly with respect to x and using the Chain Rule, we obtain $f(g(x)) = x \quad\Rightarrow\quad f'(g(x))\,g'(x) = 1 \quad\Rightarrow\quad g'(x) = \dfrac{1}{f'(g(x))}$. Using the second given equation to expand the denominator of this expression gives $g'(x) = \dfrac{1}{1+[f(g(x))]^2}$. But the first given equation states that $f(g(x)) = x$, so $g'(x) = \dfrac{1}{1+x^2}$.

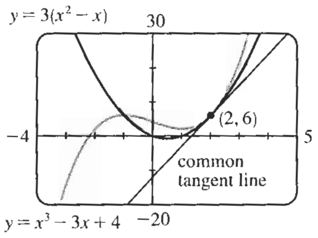
111. $\dfrac{d}{dx}[f(2x)] = x^2 \quad\Rightarrow\quad f'(2x)\cdot 2 = x^2 \quad\Rightarrow\quad f'(2x) = \frac{1}{2}x^2$. Let $t = 2x$. Then $f'(t) = \frac{1}{2}\left(\frac{1}{2}t\right)^2 = \frac{1}{8}t^2$, so $f'(x) = \frac{1}{8}x^2$.

112. Let (b, c) be on the curve, that is, $b^{2/3} + c^{2/3} = a^{2/3}$. Now $x^{2/3} + y^{2/3} = a^{2/3} \quad\Rightarrow\quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{y^{1/3}}{x^{1/3}} = -\left(\dfrac{y}{x}\right)^{1/3}$, so at (b, c) the slope of the tangent line is $-(c/b)^{1/3}$ and an equation of the tangent line is $y - c = -(c/b)^{1/3}(x-b)$ or $y = -(c/b)^{1/3}x + (c + b^{2/3}c^{1/3})$. Setting $y = 0$, we find that the x-intercept is $b^{1/3}c^{2/3} + b = b^{1/3}(c^{2/3} + b^{2/3}) = b^{1/3}a^{2/3}$ and setting $x = 0$ we find that the y-intercept is $c + b^{2/3}c^{1/3} = c^{1/3}(c^{2/3} + b^{2/3}) = c^{1/3}a^{2/3}$. So the length of the tangent line between these two points is
$$\sqrt{(b^{1/3}a^{2/3})^2 + (c^{1/3}a^{2/3})^2} = \sqrt{b^{2/3}a^{4/3} + c^{2/3}a^{4/3}} = \sqrt{(b^{2/3}+c^{2/3})a^{4/3}}$$
$$= \sqrt{a^{2/3}a^{4/3}} = \sqrt{a^2} = a = \text{constant}$$

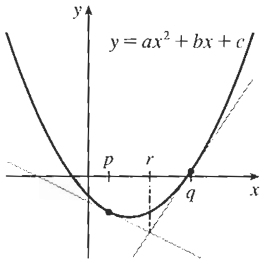
☐ PROBLEMS PLUS

1. Let a be the x-coordinate of Q. Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3} \;\Rightarrow\; a = \frac{\sqrt{3}}{2}$. Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

2. $y = x^3 - 3x + 4 \;\Rightarrow\; y' = 3x^2 - 3$, and $y = 3(x^2 - x) \;\Rightarrow\; y' = 6x - 3$. The slopes of the tangents of the two curves are equal when $3x^2 - 3 = 6x - 3$; that is, when $x = 0$ or 2. At $x = 0$, both tangents have slope -3, but the curves do not intersect. At $x = 2$, both tangents have slope 9 and the curves intersect at $(2, 6)$. So there is a common tangent line at $(2, 6)$, $y = 9x - 12$.



3.



We must show that r (in the figure) is halfway between p and q, that is, $r = (p + q)/2$. For the parabola $y = ax^2 + bx + c$, the slope of the tangent line is given by $y' = 2ax + b$. An equation of the tangent line at $x = p$ is $y - (ap^2 + bp + c) = (2ap + b)(x - p)$. Solving for y gives us

$$y = (2ap + b)x - 2ap^2 - bp + (ap^2 + bp + c)$$

or

$$y = (2ap + b)x + c - ap^2 \qquad \textbf{(1)}$$

Similarly, an equation of the tangent line at $x = q$ is

$$y = (2aq + b)x + c - aq^2 \qquad \textbf{(2)}$$

We can eliminate y and solve for x by subtracting equation **(1)** from equation **(2)**.

$$\begin{aligned}[(2aq + b) - (2ap + b)]x - aq^2 + ap^2 &= 0\\ (2aq - 2ap)x &= aq^2 - ap^2\\ 2a(q - p)x &= a(q^2 - p^2)\\ x &= \frac{a(q + p)(q - p)}{2a(q - p)} = \frac{p + q}{2}\end{aligned}$$

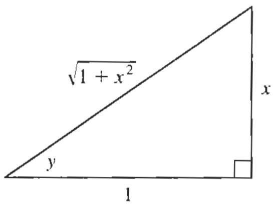
Thus, the x-coordinate of the point of intersection of the two tangent lines, namely r, is $(p + q)/2$.

4. We could differentiate and then simplify or we can simplify and then differentiate. The latter seems to be the simpler method.

$$\begin{aligned}\frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x} &= \frac{\sin^2 x}{1 + \dfrac{\cos x}{\sin x}} \cdot \frac{\sin x}{\sin x} + \frac{\cos^2 x}{1 + \dfrac{\sin x}{\cos x}} \cdot \frac{\cos x}{\cos x} = \frac{\sin^3 x}{\sin x + \cos x} + \frac{\cos^3 x}{\cos x + \sin x}\\ &= \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} \quad [\text{factor sum of cubes}] \quad = \frac{(\sin x + \cos x)(\sin^2 x - \sin x\,\cos x + \cos^2 x)}{\sin x + \cos x}\\ &= \sin^2 x - \sin x\,\cos x + \cos^2 x = 1 - \sin x\,\cos x = 1 - \tfrac{1}{2}(2\sin x\,\cos x) = 1 - \tfrac{1}{2}\sin 2x\end{aligned}$$

Thus, $\dfrac{d}{dx}\left(\dfrac{\sin^2 x}{1 + \cot x} + \dfrac{\cos^2 x}{1 + \tan x}\right) = \dfrac{d}{dx}\left(1 - \tfrac{1}{2}\sin 2x\right) = -\tfrac{1}{2}\cos 2x \cdot 2 = -\cos 2x$.

5. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that



$\sin(\tan^{-1} x) = \sin y = \dfrac{x}{\sqrt{1+x^2}}$. Using this fact we have that

$$\sin(\tan^{-1}(\sinh x)) = \frac{\sinh x}{\sqrt{1+\sinh^2 x}} = \frac{\sinh x}{\cosh x} = \tanh x.$$

Hence, $\sin^{-1}(\tanh x) = \sin^{-1}(\sin(\tan^{-1}(\sinh x))) = \tan^{-1}(\sinh x)$.

6. We find the equation of the parabola by substituting the point $(-100, 100)$, at which the car is situated, into the general equation $y = ax^2$: $100 = a(-100)^2 \Rightarrow a = \frac{1}{100}$. Now we find the equation of a tangent to the parabola at the point (x_0, y_0). We can show that $y' = a(2x) = \frac{1}{100}(2x) = \frac{1}{50}x$, so an equation of the tangent is $y - y_0 = \frac{1}{50}x_0(x - x_0)$. Since the point (x_0, y_0) is on the parabola, we must have $y_0 = \frac{1}{100}x_0^2$, so our equation of the tangent can be simplified to $y = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(x - x_0)$. We want the statue to be located on the tangent line, so we substitute its coordinates $(100, 50)$ into this equation: $50 = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(100 - x_0) \Rightarrow x_0^2 - 200x_0 + 5000 = 0 \Rightarrow$ $x_0 = \frac{1}{2}\left[200 \pm \sqrt{200^2 - 4(5000)}\right] \Rightarrow x_0 = 100 \pm 50\sqrt{2}$. But $x_0 < 100$, so the car's headlights illuminate the statue when it is located at the point $\left(100 - 50\sqrt{2}, 150 - 100\sqrt{2}\right) \approx (29.3, 8.6)$, that is, about 29.3 m east and 8.6 m north of the origin.

7. We use mathematical induction. Let S_n be the statement that $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos(4x + n\pi/2)$.

S_1 is true because

$$\begin{aligned}\frac{d}{dx}(\sin^4 x + \cos^4 x) &= 4\sin^3 x\,\cos x - 4\cos^3 x\,\sin x = 4\sin x\,\cos x\left(\sin^2 x - \cos^2 x\right)x \\ &= -4\sin x\,\cos x\,\cos 2x = -2\sin 2x\,\cos 2 = -\sin 4x = \sin(-4x) \\ &= \cos\left(\tfrac{\pi}{2} - (-4x)\right) = \cos\left(\tfrac{\pi}{2} + 4x\right) = 4^{n-1}\cos\left(4x + n\tfrac{\pi}{2}\right) \text{ when } n = 1\end{aligned}$$

Now assume S_k is true, that is, $\dfrac{d^k}{dx^k}\left(\sin^4 x + \cos^4 x\right) = 4^{k-1}\cos\left(4x + k\frac{\pi}{2}\right)$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}(\sin^4 x + \cos^4 x) &= \frac{d}{dx}\left[\frac{d^k}{dx^k}(\sin^4 x + \cos^4 x)\right] = \frac{d}{dx}\left[4^{k-1}\cos\left(4x + k\tfrac{\pi}{2}\right)\right] \\ &= -4^{k-1}\sin\left(4x + k\tfrac{\pi}{2}\right)\cdot\frac{d}{dx}\left(4x + k\tfrac{\pi}{2}\right) = -4^k\sin\left(4x + k\tfrac{\pi}{2}\right) \\ &= 4^k\sin\left(-4x - k\tfrac{\pi}{2}\right) = 4^k\cos\left(\tfrac{\pi}{2} - \left(-4x - k\tfrac{\pi}{2}\right)\right) = 4^k\cos\left(4x + (k+1)\tfrac{\pi}{2}\right)\end{aligned}$$

which shows that S_{k+1} is true.

Therefore, $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = 4^{n-1}\cos\left(4x + n\frac{\pi}{2}\right)$ for every positive integer n, by mathematical induction.

Another proof: First write

$$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\,\cos^2 x = 1 - \tfrac{1}{2}\sin^2 2x = 1 - \tfrac{1}{4}(1 - \cos 4x) = \tfrac{3}{4} + \tfrac{1}{4}\cos 4x$$

Then we have $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = \dfrac{d^n}{dx^n}\left(\dfrac{3}{4} + \dfrac{1}{4}\cos 4x\right) = \dfrac{1}{4}\cdot 4^n\cos\left(4x + n\dfrac{\pi}{2}\right) = 4^{n-1}\cos\left(4x + n\dfrac{\pi}{2}\right)$.

8. If we divide $1 - x$ into x^n by long division, we find that $f(x) = \dfrac{x^n}{1-x} = -x^{n-1} - x^{n-2} - \cdots - x - 1 + \dfrac{1}{1-x}$.

This can also be seen by multiplying the last expression by $1 - x$ and canceling terms on the right-hand side. So we let

$g(x) = 1 + x + x^2 + \cdots + x^{n-1}$, so that $f(x) = \dfrac{1}{1-x} - g(x) \;\Rightarrow\; f^{(n)}(x) = \left(\dfrac{1}{1-x}\right)^{(n)} - g^{(n)}(x)$.

But g is a polynomial of degree $(n-1)$, so its nth derivative is 0, and therefore $f^{(n)}(x) = \left(\dfrac{1}{1-x}\right)^{(n)}$. Now

$$\frac{d}{dx}(1-x)^{-1} = (-1)(1-x)^{-2}(-1) = (1-x)^{-2},\quad \frac{d^2}{dx^2}(1-x)^{-1} = (-2)(1-x)^{-3}(-1) = 2(1-x)^{-3},$$

$$\frac{d^3}{dx^3}(1-x)^{-1} = (-3)\cdot 2(1-x)^{-4}(-1) = 3\cdot 2(1-x)^{-4},\quad \frac{d^4}{dx^4}(1-x)^{-1} = 4\cdot 3\cdot 2(1-x)^{-5}, \text{ and so on.}$$

So after n differentiations, we will have $f^{(n)}(x) = \left(\dfrac{1}{1-x}\right)^{(n)} = \dfrac{n!}{(1-x)^{n+1}}$.

9. We must find a value x_0 such that the normal lines to the parabola $y = x^2$ at $x = \pm x_0$ intersect at a point one unit from the points $\left(\pm x_0, x_0^2\right)$. The normals to $y = x^2$ at $x = \pm x_0$ have slopes $-\dfrac{1}{\pm 2x_0}$ and pass through $\left(\pm x_0, x_0^2\right)$ respectively, so the normals have the equations $y - x_0^2 = -\dfrac{1}{2x_0}(x - x_0)$ and $y - x_0^2 = \dfrac{1}{2x_0}(x + x_0)$. The common y-intercept is $x_0^2 + \dfrac{1}{2}$.

We want to find the value of x_0 for which the distance from $\left(0, x_0^2 + \frac{1}{2}\right)$ to $\left(x_0, x_0^2\right)$ equals 1. The square of the distance is $(x_0 - 0)^2 + \left[x_0^2 - \left(x_0^2 + \frac{1}{2}\right)\right]^2 = x_0^2 + \frac{1}{4} = 1 \;\Leftrightarrow\; x_0 = \pm\frac{\sqrt{3}}{2}$. For these values of x_0, the y-intercept is $x_0^2 + \frac{1}{2} = \frac{5}{4}$, so the center of the circle is at $\left(0, \frac{5}{4}\right)$.

Another solution: Let the center of the circle be $(0, a)$. Then the equation of the circle is $x^2 + (y - a)^2 = 1$.

Solving with the equation of the parabola, $y = x^2$, we get $x^2 + (x^2 - a)^2 = 1 \;\Leftrightarrow\; x^2 + x^4 - 2ax^2 + a^2 = 1 \;\Leftrightarrow$ $x^4 + (1 - 2a)x^2 + a^2 - 1 = 0$. The parabola and the circle will be tangent to each other when this quadratic equation in x^2 has equal roots; that is, when the discriminant is 0. Thus, $(1 - 2a)^2 - 4(a^2 - 1) = 0 \;\Leftrightarrow$ $1 - 4a + 4a^2 - 4a^2 + 4 = 0 \;\Leftrightarrow\; 4a = 5$, so $a = \frac{5}{4}$. The center of the circle is $\left(0, \frac{5}{4}\right)$.

10.
$$\lim_{x\to a}\frac{f(x)-f(a)}{\sqrt{x}-\sqrt{a}} = \lim_{x\to a}\left[\frac{f(x)-f(a)}{\sqrt{x}-\sqrt{a}}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right] = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\cdot\left(\sqrt{x}+\sqrt{a}\right)\right]$$
$$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}\left(\sqrt{x}+\sqrt{a}\right) = f'(a)\cdot\left(\sqrt{a}+\sqrt{a}\right) = 2\sqrt{a}\,f'(a)$$

11. We can assume without loss of generality that $\theta = 0$ at time $t = 0$, so that $\theta = 12\pi t$ rad. [The angular velocity of the wheel is 360 rpm $= 360\cdot(2\pi \text{ rad})/(60 \text{ s}) = 12\pi$ rad/s.] Then the position of A as a function of time is

$A = (40\cos\theta, 40\sin\theta) = (40\cos 12\pi t, 40\sin 12\pi t)$, so $\sin\alpha = \dfrac{y}{1.2\text{ m}} = \dfrac{40\sin\theta}{120} = \dfrac{\sin\theta}{3} = \dfrac{1}{3}\sin 12\pi t$.

(a) Differentiating the expression for $\sin\alpha$, we get $\cos\alpha\cdot\dfrac{d\alpha}{dt} = \dfrac{1}{3}\cdot 12\pi\cdot\cos 12\pi t = 4\pi\cos\theta$. When $\theta = \dfrac{\pi}{3}$, we have

$$\sin\alpha = \frac{1}{3}\sin\theta = \frac{\sqrt{3}}{6}, \text{ so } \cos\alpha = \sqrt{1 - \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{11}{12}} \text{ and } \frac{d\alpha}{dt} = \frac{4\pi\cos\frac{\pi}{3}}{\cos\alpha} = \frac{2\pi}{\sqrt{11/12}} = \frac{4\pi\sqrt{3}}{\sqrt{11}} \approx 6.56 \text{ rad/s}.$$

(b) By the Law of Cosines, $|AP|^2 = |OA|^2 + |OP|^2 - 2\,|OA|\,|OP|\cos\theta \;\Rightarrow$

$$120^2 = 40^2 + |OP|^2 - 2\cdot 40\,|OP|\cos\theta \;\Rightarrow\; |OP|^2 - (80\cos\theta)\,|OP| - 12{,}800 = 0 \;\Rightarrow$$

$$|OP| = \tfrac{1}{2}\left(80\cos\theta \pm \sqrt{6400\cos^2\theta + 51{,}200}\right) = 40\cos\theta \pm 40\sqrt{\cos^2\theta + 8} = 40\left(\cos\theta + \sqrt{8+\cos^2\theta}\right)\text{ cm}$$

[since $|OP| > 0$]. As a check, note that $|OP| = 160$ cm when $\theta = 0$ and $|OP| = 80\sqrt{2}$ cm when $\theta = \frac{\pi}{2}$.

(c) By part (b), the x-coordinate of P is given by $x = 40\left(\cos\theta + \sqrt{8+\cos^2\theta}\right)$, so

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = 40\left(-\sin\theta - \frac{2\cos\theta\sin\theta}{2\sqrt{8+\cos^2\theta}}\right)\cdot 12\pi = -480\pi\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{8+\cos^2\theta}}\right)\text{ cm/s.}$$

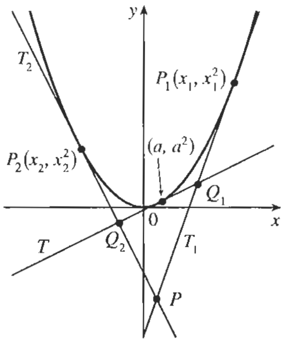
In particular, $dx/dt = 0$ cm/s when $\theta = 0$ and $dx/dt = -480\pi$ cm/s when $\theta = \frac{\pi}{2}$.

12. The equation of T_1 is $y - x_1^2 = 2x_1(x - x_1) = 2x_1x - 2x_1^2$ or $y = 2x_1x - x_1^2$. The equation of T_2 is $y = 2x_2x - x_2^2$. Solving for the point of intersection, we get $2x(x_1 - x_2) = x_1^2 - x_2^2 \;\Rightarrow\; x = \frac{1}{2}(x_1 + x_2)$. Therefore, the coordinates of P are $\left(\frac{1}{2}(x_1 + x_2), x_1x_2\right)$. So if the point of contact of T is (a, a^2), then Q_1 is $\left(\frac{1}{2}(a + x_1), ax_1\right)$ and Q_2 is $\left(\frac{1}{2}(a + x_2), ax_2\right)$. Therefore,

$$|PQ_1|^2 = \tfrac{1}{4}(a - x_2)^2 + x_1^2(a - x_2)^2 = (a - x_2)^2\left(\tfrac{1}{4} + x_1^2\right) \text{ and}$$

$$|PP_1|^2 = \tfrac{1}{4}(x_1 - x_2)^2 + x_1^2(x_1 - x_2)^2 = (x_1 - x_2)^2\left(\tfrac{1}{4} + x_1^2\right).$$

So $\dfrac{|PQ_1|^2}{|PP_1|^2} = \dfrac{(a - x_2)^2}{(x_1 - x_2)^2}$, and similarly $\dfrac{|PQ_2|^2}{|PP_2|^2} = \dfrac{(x_1 - a)^2}{(x_1 - x_2)^2}$. Finally, $\dfrac{|PQ_1|}{|PP_1|} + \dfrac{|PQ_2|}{|PP_2|} = \dfrac{a - x_2}{x_1 - x_2} + \dfrac{x_1 - a}{x_1 - x_2} = 1$.



13. Consider the statement that $\dfrac{d^n}{dx^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx + n\theta)$. For $n = 1$,

$$\frac{d}{dx}(e^{ax}\sin bx) = ae^{ax}\sin bx + be^{ax}\cos bx, \text{ and}$$

$$re^{ax}\sin(bx + \theta) = re^{ax}[\sin bx\cos\theta + \cos bx\sin\theta] = re^{ax}\left(\frac{a}{r}\sin bx + \tfrac{b}{r}\cos bx\right) = ae^{ax}\sin bx + be^{ax}\cos bx$$

since $\tan\theta = \dfrac{b}{a} \;\Rightarrow\; \sin\theta = \dfrac{b}{r}$ and $\cos\theta = \dfrac{a}{r}$. So the statement is true for $n = 1$.

Assume it is true for $n = k$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) &= \frac{d}{dx}\left[r^k e^{ax}\sin(bx + k\theta)\right] = r^k ae^{ax}\sin(bx + k\theta) + r^k e^{ax} b\cos(bx + k\theta)\\ &= r^k e^{ax}[a\sin(bx + k\theta) + b\cos(bx + k\theta)]\end{aligned}$$

But

$$\sin[bx + (k+1)\theta] = \sin[(bx + k\theta) + \theta] = \sin(bx + k\theta)\cos\theta + \sin\theta\cos(bx + k\theta) = \tfrac{a}{r}\sin(bx + k\theta) + \tfrac{b}{r}\cos(bx + k\theta).$$

Hence, $a\sin(bx + k\theta) + b\cos(bx + k\theta) = r\sin[bx + (k+1)\theta]$. So

$$\frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) = r^k e^{ax}[a\sin(bx + k\theta) + b\cos(bx + k\theta)] = r^k e^{ax}[r\sin(bx + (k+1)\theta)] = r^{k+1}e^{ax}[\sin(bx + (k+1)\theta)].$$

Therefore, the statement is true for all n by mathematical induction.

14. We recognize this limit as the definition of the derivative of the function $f(x) = e^{\sin x}$ at $x = \pi$, since it is of the form $\lim\limits_{x \to \pi} \dfrac{f(x) - f(\pi)}{x - \pi}$. Therefore, the limit is equal to $f'(\pi) = (\cos \pi) e^{\sin \pi} = -1 \cdot e^0 = -1$.

15. It seems from the figure that as P approaches the point $(0, 2)$ from the right, $x_T \to \infty$ and $y_T \to 2^+$. As P approaches the point $(3, 0)$ from the left, it appears that $x_T \to 3^+$ and $y_T \to \infty$. So we guess that $x_T \in (3, \infty)$ and $y_T \in (2, \infty)$. It is more difficult to estimate the range of values for x_N and y_N. We might perhaps guess that $x_N \in (0, 3)$, and $y_N \in (-\infty, 0)$ or $(-2, 0)$.

In order to actually solve the problem, we implicitly differentiate the equation of the ellipse to find the equation of the tangent line: $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1 \;\Rightarrow\; \dfrac{2x}{9} + \dfrac{2y}{4} y' = 0$, so $y' = -\dfrac{4}{9} \dfrac{x}{y}$. So at the point (x_0, y_0) on the ellipse, an equation of the tangent line is $y - y_0 = -\dfrac{4}{9} \dfrac{x_0}{y_0}(x - x_0)$ or $4x_0 x + 9y_0 y = 4x_0^2 + 9y_0^2$. This can be written as $\dfrac{x_0 x}{9} + \dfrac{y_0 y}{4} = \dfrac{x_0^2}{9} + \dfrac{y_0^2}{4} = 1$, because (x_0, y_0) lies on the ellipse. So an equation of the tangent line is $\dfrac{x_0 x}{9} + \dfrac{y_0 y}{4} = 1$.

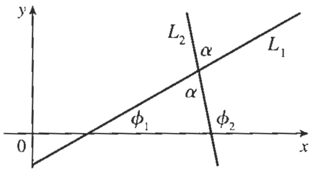
Therefore, the x-intercept x_T for the tangent line is given by $\dfrac{x_0 x_T}{9} = 1 \;\Leftrightarrow\; x_T = \dfrac{9}{x_0}$, and the y-intercept y_T is given by $\dfrac{y_0 y_T}{4} = 1 \;\Leftrightarrow\; y_T = \dfrac{4}{y_0}$.

So as x_0 takes on all values in $(0, 3)$, x_T takes on all values in $(3, \infty)$, and as y_0 takes on all values in $(0, 2)$, y_T takes on all values in $(2, \infty)$. At the point (x_0, y_0) on the ellipse, the slope of the normal line is $-\dfrac{1}{y'(x_0, y_0)} = \dfrac{9}{4} \dfrac{y_0}{x_0}$, and its equation is $y - y_0 = \dfrac{9}{4} \dfrac{y_0}{x_0}(x - x_0)$. So the x-intercept x_N for the normal line is given by $0 - y_0 = \dfrac{9}{4} \dfrac{y_0}{x_0}(x_N - x_0) \;\Rightarrow\; x_N = -\dfrac{4x_0}{9} + x_0 = \dfrac{5x_0}{9}$, and the y-intercept y_N is given by $y_N - y_0 = \dfrac{9}{4} \dfrac{y_0}{x_0}(0 - x_0) \;\Rightarrow\; y_N = -\dfrac{9y_0}{4} + y_0 = -\dfrac{5y_0}{4}$.

So as x_0 takes on all values in $(0, 3)$, x_N takes on all values in $\left(0, \frac{5}{3}\right)$, and as y_0 takes on all values in $(0, 2)$, y_N takes on all values in $\left(-\frac{5}{2}, 0\right)$.

16. $\lim\limits_{x \to 0} \dfrac{\sin(3 + x)^2 - \sin 9}{x} = f'(3)$ where $f(x) = \sin x^2$. Now $f'(x) = (\cos x^2)(2x)$, so $f'(3) = 6 \cos 9$.

17. (a) If the two lines L_1 and L_2 have slopes m_1 and m_2 and angles of inclination ϕ_1 and ϕ_2, then $m_1 = \tan \phi_1$ and $m_2 = \tan \phi_2$. The triangle in the figure shows that $\phi_1 + \alpha + (180° - \phi_2) = 180°$ and so $\alpha = \phi_2 - \phi_1$. Therefore, using the identity for $\tan(x - y)$, we have $\tan \alpha = \tan(\phi_2 - \phi_1) = \dfrac{\tan \phi_2 - \tan \phi_1}{1 + \tan \phi_2 \tan \phi_1}$ and so $\tan \alpha = \dfrac{m_2 - m_1}{1 + m_1 m_2}$.



(b) (i) The parabolas intersect when $x^2 = (x-2)^2 \Rightarrow x = 1$. If $y = x^2$, then $y' = 2x$, so the slope of the tangent to $y = x^2$ at $(1,1)$ is $m_1 = 2(1) = 2$. If $y = (x-2)^2$, then $y' = 2(x-2)$, so the slope of the tangent to $y = (x-2)^2$ at $(1,1)$ is $m_2 = 2(1-2) = -2$. Therefore, $\tan\alpha = \dfrac{m_2 - m_1}{1 + m_1 m_2} = \dfrac{-2-2}{1+2(-2)} = \dfrac{4}{3}$ and so $\alpha = \tan^{-1}\left(\frac{4}{3}\right) \approx 53^\circ$ [or 127°].

(ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$ intersect when $x^2 - 4x + (x^2 - 3) + 3 = 0 \Leftrightarrow 2x(x-2) = 0 \Rightarrow$ $x = 0$ or 2, but 0 is extraneous. If $x = 2$, then $y = \pm 1$. If $x^2 - y^2 = 3$ then $2x - 2yy' = 0 \Rightarrow y' = x/y$ and $x^2 - 4x + y^2 + 3 = 0 \Rightarrow 2x - 4 + 2yy' = 0 \Rightarrow y' = \dfrac{2-x}{y}$. At $(2,1)$ the slopes are $m_1 = 2$ and $m_2 = 0$, so $\tan\alpha = \frac{0-2}{1+2\cdot 0} = -2 \Rightarrow \alpha \approx 117^\circ$. At $(2,-1)$ the slopes are $m_1 = -2$ and $m_2 = 0$, so $\tan\alpha = \dfrac{0-(-2)}{1+(-2)(0)} = 2 \Rightarrow \alpha \approx 63^\circ$ [or 117°].

18. $y^2 = 4px \Rightarrow 2yy' = 4p \Rightarrow y' = 2p/y \Rightarrow$ slope of tangent at $P(x_1, y_1)$ is $m_1 = 2p/y_1$. The slope of FP is $m_2 = \dfrac{y_1}{x_1 - p}$, so by the formula from Problem 17(a),

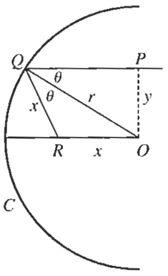
$$\tan\alpha = \frac{\dfrac{y_1}{x_1-p} - \dfrac{2p}{y_1}}{1 + \left(\dfrac{2p}{y_1}\right)\left(\dfrac{y_1}{x_1-p}\right)} \cdot \frac{y_1(x_1-p)}{y_1(x_1-p)} = \frac{y_1^2 - 2p(x_1-p)}{y_1(x_1-p) + 2py_1}$$

$$= \frac{4px_1 - 2px_1 + 2p^2}{x_1y_1 - py_1 + 2py_1} = \frac{2p(p+x_1)}{y_1(p+x_1)} = \frac{2p}{y_1}$$

$$= \text{slope of tangent at } P = \tan\beta$$

Since $0 \le \alpha, \beta \le \frac{\pi}{2}$, this proves that $\alpha = \beta$.

19. Since $\angle ROQ = \angle OQP = \theta$, the triangle QOR is isosceles, so $|QR| = |RO| = x$. By the Law of Cosines, $x^2 = x^2 + r^2 - 2rx\cos\theta$. Hence, $2rx\cos\theta = r^2$, so $x = \dfrac{r^2}{2r\cos\theta} = \dfrac{r}{2\cos\theta}$. Note that as $y \to 0^+$, $\theta \to 0^+$ (since $\sin\theta = y/r$), and hence $x \to \dfrac{r}{2\cos 0} = \dfrac{r}{2}$. Thus, as P is taken closer and closer to the x-axis, the point R approaches the midpoint of the radius AO.



20. $$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f(x)-0}{g(x)-0} = \lim_{x\to 0}\frac{f(x)-f(0)}{g(x)-g(0)} = \lim_{x\to 0}\frac{\dfrac{f(x)-f(0)}{x-0}}{\dfrac{g(x)-g(0)}{x-0}} = \frac{\displaystyle\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}}{\displaystyle\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}} = \frac{f'(0)}{g'(0)}$$

21. $$\lim_{x\to 0}\frac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin a\,\cos 2x+\cos a\,\sin 2x-2\sin a\,\cos x-2\cos a\,\sin x+\sin a}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin a\,(\cos 2x-2\cos x+1)+\cos a\,(\sin 2x-2\sin x)}{x^2}$$

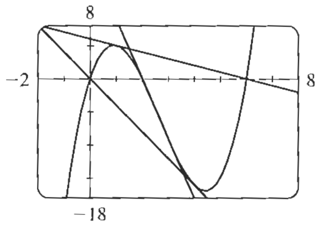
$$=\lim_{x\to 0}\frac{\sin a\,(2\cos^2 x-1-2\cos x+1)+\cos a\,(2\sin x\,\cos x-2\sin x)}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin a\,(2\cos x)(\cos x-1)+\cos a\,(2\sin x)(\cos x-1)}{x^2}$$

$$=\lim_{x\to 0}\frac{2(\cos x-1)[\sin a\,\cos x+\cos a\,\sin x](\cos x+1)}{x^2(\cos x+1)}$$

$$=\lim_{x\to 0}\frac{-2\sin^2 x\,[\sin(a+x)]}{x^2(\cos x+1)}=-2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{\sin(a+x)}{\cos x+1}=-2(1)^2\frac{\sin(a+0)}{\cos 0+1}=-\sin a$$

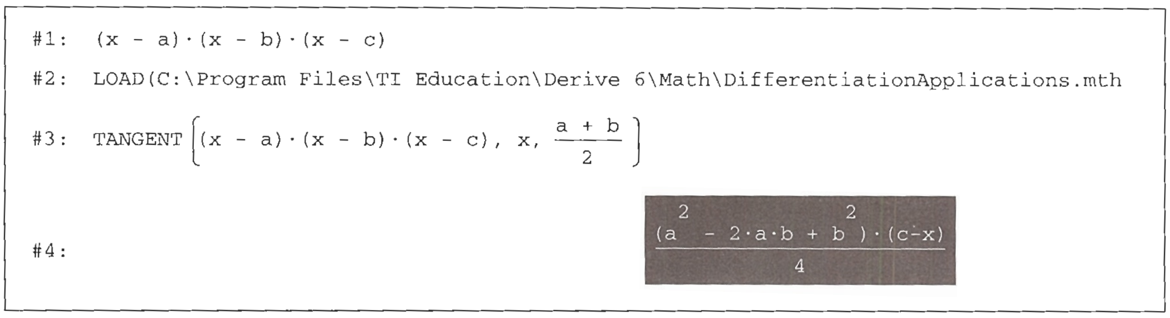
22. (a) $f(x)=x(x-2)(x-6)=x^3-8x^2+12x \Rightarrow$ $f'(x)=3x^2-16x+12$. The average of the first pair of zeros is $(0+2)/2=1$. At $x=1$, the slope of the tangent line is $f'(1)=-1$, so an equation of the tangent line has the form $y=-1x+b$. Since $f(1)=5$, we have $5=-1+b \Rightarrow b=6$ and the tangent has equation $y=-x+6$.



Similarly, at $x=\dfrac{0+6}{2}=3$, $y=-9x+18$; at $x=\dfrac{2+6}{2}=4$, $y=-4x$. From the graph, we see that each tangent line drawn at the average of two zeros intersects the graph of f at the third zero.

(b) A CAS gives $f'(x)=(x-b)(x-c)+(x-a)(x-c)+(x-a)(x-b)$ or $f'(x)=3x^2-2(a+b+c)x+ab+ac+bc$. Using the `Simplify` command, we get $f'\left(\dfrac{a+b}{2}\right)=-\dfrac{(a-b)^2}{4}$ and $f\left(\dfrac{a+b}{2}\right)=-\dfrac{(a-b)^2}{8}(a+b-2c)$, so an equation of the tangent line at $x=\dfrac{a+b}{2}$ is $y=-\dfrac{(a-b)^2}{4}\left(x-\dfrac{a+b}{2}\right)-\dfrac{(a-b)^2}{8}(a+b-2c)$. To find the x-intercept, let $y=0$ and use the `Solve` command. The result is $x=c$.

Using Derive, we can begin by authoring the expression $(x-a)(x-b)(x-c)$. Now load the utility file `DifferentiationApplications`. Next we author tangent $(\#1, x, (a+b)/2)$—this is the command to find an equation of the tangent line of the function in #1 whose independent variable is x at the x-value $(a+b)/2$. We then simplify that expression and obtain the equation $y=\#4$. The form in expression #4 makes it easy to see that the x-intercept is the third zero, namely c. In a similar fashion we see that b is the x-intercept for the tangent line at $(a+c)/2$ and a is the x-intercept for the tangent line at $(b+c)/2$.

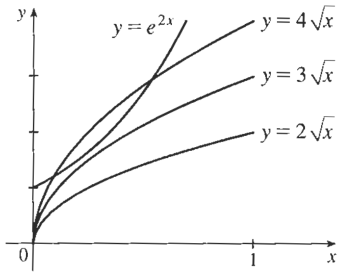


```
#1:  (x - a)·(x - b)·(x - c)

#2:  LOAD(C:\Program Files\TI Education\Derive 6\Math\DifferentiationApplications.mth
```

$$\#3:\ \ \mathrm{TANGENT}\left((x-a)\cdot(x-b)\cdot(x-c),\ x,\ \frac{a+b}{2}\right)$$

$$\#4:\ \ \frac{(a^2-2\cdot a\cdot b+b^2)\cdot(c-x)}{4}$$

23. 

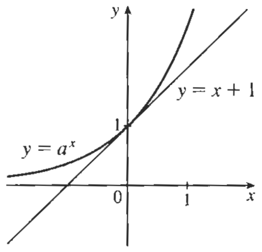
Let $f(x)=e^{2x}$ and $g(x)=k\sqrt{x}$ $[k>0]$. From the graphs of f and g, we see that f will intersect g exactly once when f and g share a tangent line. Thus, we must have $f=g$ and $f'=g'$ at $x=a$.

$$f(a)=g(a) \;\Rightarrow\; e^{2a}=k\sqrt{a} \quad (\star)$$

and $$f'(a)=g'(a) \;\Rightarrow\; 2e^{2a}=\frac{k}{2\sqrt{a}} \;\Rightarrow\; e^{2a}=\frac{k}{4\sqrt{a}}.$$

So we must have $k\sqrt{a}=\dfrac{k}{4\sqrt{a}} \;\Rightarrow\; \left(\sqrt{a}\right)^2=\dfrac{k}{4k} \;\Rightarrow\; a=\frac{1}{4}$. From $(\star)$, $e^{2(1/4)}=k\sqrt{1/4} \;\Rightarrow$ $k=2e^{1/2}=2\sqrt{e}\approx 3.297$.

24. We see that at $x=0$, $f(x)=a^x=1+x=1$, so if $y=a^x$ is to lie above $y=1+x$, the two curves must just touch at $(0,1)$, that is, we must have $f'(0)=1$. [To see this analytically, note that $a^x\geq 1+x \;\Rightarrow\; a^x-1\geq x \;\Rightarrow\; \dfrac{a^x-1}{x}\geq 1$ for $x>0$, so $f'(0)=\displaystyle\lim_{x\to 0^+}\frac{a^x-1}{x}\geq 1$. Similarly, for $x<0$, $a^x-1\geq x \;\Rightarrow\; \dfrac{a^x-1}{x}\leq 1$, so $f'(0)=\displaystyle\lim_{x\to 0^-}\frac{a^x-1}{x}\leq 1$.



Since $1\leq f'(0)\leq 1$, we must have $f'(0)=1$.] But $f'(x)=a^x\ln a \;\Rightarrow\; f'(0)=\ln a$, so we have $\ln a=1 \;\Leftrightarrow\; a=e$.

Another method: The inequality certainly holds for $x\leq -1$, so consider $x>-1$, $x\neq 0$. Then $a^x\geq 1+x \;\Rightarrow$ $a\geq(1+x)^{1/x}$ for $x>0 \;\Rightarrow\; a\geq\displaystyle\lim_{x\to 0^+}(1+x)^{1/x}=e$, by Equation 3.6.5. Also, $a^x\geq 1+x \;\Rightarrow\; a\leq(1+x)^{1/x}$ for $x<0 \;\Rightarrow\; a\leq\displaystyle\lim_{x\to 0^-}(1+x)^{1/x}=e$. So since $e\leq a\leq e$, we must have $a=e$.

25. $y=\dfrac{x}{\sqrt{a^2-1}}-\dfrac{2}{\sqrt{a^2-1}}\arctan\dfrac{\sin x}{a+\sqrt{a^2-1}+\cos x}$. Let $k=a+\sqrt{a^2-1}$. Then

$$\begin{aligned}
y'&=\frac{1}{\sqrt{a^2-1}}-\frac{2}{\sqrt{a^2-1}}\cdot\frac{1}{1+\sin^2 x/(k+\cos x)^2}\cdot\frac{\cos x(k+\cos x)+\sin^2 x}{(k+\cos x)^2}\\
&=\frac{1}{\sqrt{a^2-1}}-\frac{2}{\sqrt{a^2-1}}\cdot\frac{k\cos x+\cos^2 x+\sin^2 x}{(k+\cos x)^2+\sin^2 x}=\frac{1}{\sqrt{a^2-1}}-\frac{2}{\sqrt{a^2-1}}\cdot\frac{k\cos x+1}{k^2+2k\cos x+1}\\
&=\frac{k^2+2k\cos x+1-2k\cos x-2}{\sqrt{a^2-1}\,(k^2+2k\cos x+1)}=\frac{k^2-1}{\sqrt{a^2-1}\,(k^2+2k\cos x+1)}
\end{aligned}$$

But $k^2=2a^2+2a\sqrt{a^2-1}-1=2a\left(a+\sqrt{a^2-1}\right)-1=2ak-1$, so $k^2+1=2ak$, and $k^2-1=2(ak-1)$.

So $y'=\dfrac{2(ak-1)}{\sqrt{a^2-1}\,(2ak+2k\cos x)}=\dfrac{ak-1}{\sqrt{a^2-1}k\,(a+\cos x)}$. But $ak-1=a^2+a\sqrt{a^2-1}-1=k\sqrt{a^2-1}$, so $y'=1/(a+\cos x)$.

26. Suppose that $y=mx+c$ is a tangent line to the ellipse. Then it intersects the ellipse at only one point, so the discriminant of the equation $\dfrac{x^2}{a^2}+\dfrac{(mx+c)^2}{b^2}=1 \;\Leftrightarrow\; (b^2+a^2m^2)x^2+2mca^2x+a^2c^2-a^2b^2=0$ must be 0; that is,

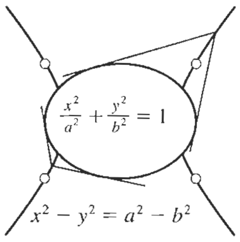
$$0 = (2mca^2)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 4a^4c^2m^2 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4c^2m^2 + 4a^4b^2m^2$$
$$= 4a^2b^2(a^2m^2 + b^2 - c^2)$$

Therefore, $a^2m^2 + b^2 - c^2 = 0$.

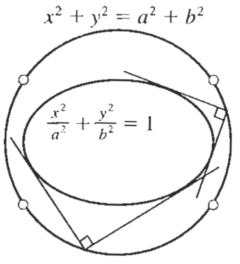
Now if a point (α, β) lies on the line $y = mx + c$, then $c = \beta - m\alpha$, so from above,

$$0 = a^2m^2 + b^2 - (\beta - m\alpha)^2 = (a^2 - \alpha^2)m^2 + 2\alpha\beta m + b^2 - \beta^2 \quad \Leftrightarrow \quad m^2 + \frac{2\alpha\beta}{a^2 - \alpha^2}m + \frac{b^2 - \beta^2}{a^2 - \alpha^2} = 0.$$

(a) Suppose that the two tangent lines from the point (α, β) to the ellipse have slopes m and $\frac{1}{m}$. Then m and $\frac{1}{m}$ are roots of the equation $z^2 + \frac{2\alpha\beta}{a^2 - \alpha^2}z + \frac{b^2 - \beta^2}{a^2 - \alpha^2} = 0$. This implies that $(z - m)\left(z - \frac{1}{m}\right) = 0 \quad \Leftrightarrow$ $z^2 - \left(m + \frac{1}{m}\right)z + m\left(\frac{1}{m}\right) = 0$, so equating the constant terms in the two quadratic equations, we get $\frac{b^2 - \beta^2}{a^2 - \alpha^2} = m\left(\frac{1}{m}\right) = 1$, and hence $b^2 - \beta^2 = a^2 - \alpha^2$. So (α, β) lies on the hyperbola $x^2 - y^2 = a^2 - b^2$.



(b) If the two tangent lines from the point (α, β) to the ellipse have slopes m and $-\frac{1}{m}$, then m and $-\frac{1}{m}$ are roots of the quadratic equation, and so $(z - m)\left(z + \frac{1}{m}\right) = 0$, and equating the constant terms as in part (a), we get $\frac{b^2 - \beta^2}{a^2 - \alpha^2} = -1$, and hence $b^2 - \beta^2 = \alpha^2 - a^2$. So the point (α, β) lies on the circle $x^2 + y^2 = a^2 + b^2$.



27. $y = x^4 - 2x^2 - x \quad \Rightarrow \quad y' = 4x^3 - 4x - 1$. The equation of the tangent line at $x = a$ is $y - (a^4 - 2a^2 - a) = (4a^3 - 4a - 1)(x - a)$ or $y = (4a^3 - 4a - 1)x + \left(-3a^4 + 2a^2\right)$ and similarly for $x = b$. So if at $x = a$ and $x = b$ we have the same tangent line, then $4a^3 - 4a - 1 = 4b^3 - 4b - 1$ and $-3a^4 + 2a^2 = -3b^4 + 2b^2$. The first equation gives $a^3 - b^3 = a - b \quad \Rightarrow \quad (a - b)(a^2 + ab + b^2) = (a - b)$. Assuming $a \neq b$, we have $1 = a^2 + ab + b^2$. The second equation gives $3(a^4 - b^4) = 2(a^2 - b^2) \quad \Rightarrow \quad 3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ which is true if $a = -b$. Substituting into $1 = a^2 + ab + b^2$ gives $1 = a^2 - a^2 + a^2 \quad \Rightarrow \quad a = \pm 1$ so that $a = 1$ and $b = -1$ or vice versa. Thus, the points $(1, -2)$ and $(-1, 0)$ have a common tangent line.

As long as there are only two such points, we are done. So we show that these are in fact the only two such points. Suppose that $a^2 - b^2 \neq 0$. Then $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ gives $3(a^2 + b^2) = 2$ or $a^2 + b^2 = \frac{2}{3}$. Thus, $ab = (a^2 + ab + b^2) - (a^2 + b^2) = 1 - \frac{2}{3} = \frac{1}{3}$, so $b = \frac{1}{3a}$. Hence, $a^2 + \frac{1}{9a^2} = \frac{2}{3}$, so $9a^4 + 1 = 6a^2 \quad \Rightarrow$ $0 = 9a^4 - 6a^2 + 1 = (3a^2 - 1)^2$. So $3a^2 - 1 = 0 \quad \Rightarrow \quad a^2 = \frac{1}{3} \quad \Rightarrow \quad b^2 = \frac{1}{9a^2} = \frac{1}{3} = a^2$, contradicting our assumption that $a^2 \neq b^2$.

28. Suppose that the normal lines at the three points (a_1, a_1^2), (a_2, a_2^2), and (a_3, a_3^2) intersect at a common point. Now if one of the a_i is 0 (suppose $a_1 = 0$) then by symmetry $a_2 = -a_3$, so $a_1 + a_2 + a_3 = 0$. So we can assume that none of the a_i is 0.

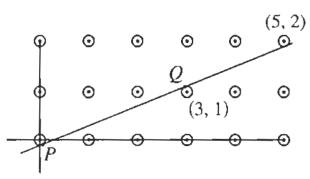
The slope of the tangent line at (a_i, a_i^2) is $2a_i$, so the slope of the normal line is $-\dfrac{1}{2a_i}$ and its equation is $y - a_i^2 = -\dfrac{1}{2a_i}(x - a_i)$. We solve for the x-coordinate of the intersection of the normal lines from (a_1, a_1^2) and (a_2, a_2^2):

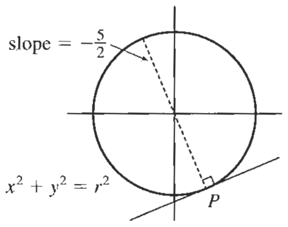
$$y = a_1^2 - \frac{1}{2a_1}(x - a_1) = a_2^2 - \frac{1}{2a_2}(x - a_2) \quad\Rightarrow\quad x\left(\frac{1}{2a_2} - \frac{1}{2a_1}\right) = a_2^2 - a_1^2 \quad\Rightarrow$$

$x\left(\dfrac{a_1 - a_2}{2a_1 a_2}\right) = (-a_1 - a_2)(a_1 + a_2) \quad\Leftrightarrow\quad x = -2a_1 a_2(a_1 + a_2)$ **(1).** Similarly, solving for the x-coordinate of the intersections of the normal lines from (a_1, a_1^2) and (a_3, a_3^2) gives $x = -2a_1 a_3(a_1 + a_3)$ **(2).**

Equating **(1)** and **(2)** gives $a_2(a_1 + a_2) = a_3(a_1 + a_3) \quad\Leftrightarrow\quad a_1(a_2 - a_3) = a_3^2 - a_2^2 = -(a_2 + a_3)(a_2 - a_3) \quad\Leftrightarrow$ $a_1 = -(a_2 + a_3) \quad\Leftrightarrow\quad a_1 + a_2 + a_3 = 0$.

29. Because of the periodic nature of the lattice points, it suffices to consider the points in the 5×2 grid shown. We can see that the minimum value of r occurs when there is a line with slope $\frac{2}{5}$ which touches the circle centered at $(3, 1)$ and the circles centered at $(0, 0)$ and $(5, 2)$.



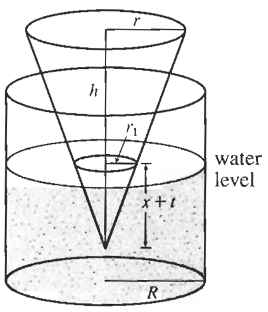


To find P, the point at which the line is tangent to the circle at $(0, 0)$, we simultaneously solve $x^2 + y^2 = r^2$ and $y = -\frac{5}{2}x \quad\Rightarrow\quad x^2 + \frac{25}{4}x^2 = r^2 \quad\Rightarrow\quad x^2 = \frac{4}{29}r^2 \quad\Rightarrow\quad x = \frac{2}{\sqrt{29}}r,\ y = -\frac{5}{\sqrt{29}}r$. To find Q, we either use symmetry or solve $(x - 3)^2 + (y - 1)^2 = r^2$ and $y - 1 = -\frac{5}{2}(x - 3)$. As above, we get $x = 3 - \frac{2}{\sqrt{29}}r$, $y = 1 + \frac{5}{\sqrt{29}}r$. Now the slope of the line PQ is $\frac{2}{5}$, so

$$m_{PQ} = \frac{1 + \frac{5}{\sqrt{29}}r - \left(-\frac{5}{\sqrt{29}}r\right)}{3 - \frac{2}{\sqrt{29}}r - \frac{2}{\sqrt{29}}r} = \frac{1 + \frac{10}{\sqrt{29}}r}{3 - \frac{4}{\sqrt{29}}r} = \frac{\sqrt{29} + 10r}{3\sqrt{29} - 4r} = \frac{2}{5} \quad\Rightarrow$$

$5\sqrt{29} + 50r = 6\sqrt{29} - 8r \quad\Leftrightarrow\quad 58r = \sqrt{29} \quad\Leftrightarrow\quad r = \frac{\sqrt{29}}{58}$. So the minimum value of r for which any line with slope $\frac{2}{5}$ intersects circles with radius r centered at the lattice points on the plane is $r = \frac{\sqrt{29}}{58} \approx 0.093$.

30.



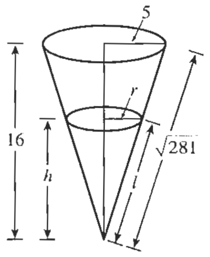
Assume the axes of the cone and the cylinder are parallel. Let H denote the initial height of the water. When the cone has been dropping for t seconds, the water level has risen x centimeters, so the tip of the cone is $x + 1t$ centimeters below the water line. We want to find dx/dt when $x + t = h$ (when the cone is completely submerged). Using similar triangles, $\dfrac{r_1}{x+t} = \dfrac{r}{h} \;\Rightarrow\; r_1 = \dfrac{r}{h}(x+t)$.

$$\begin{array}{ccccc}
\text{volume of water and cone at time } t & = & \text{original volume of water} & + & \text{volume of submerged part of cone} \\
\pi R^2(H+x) & = & \pi R^2 H & + & \frac{1}{3}\pi r_1^2(x+t) \\
\pi R^2 H + \pi R^2 x & = & \pi R^2 H & + & \frac{1}{3}\pi \frac{r^2}{h^2}(x+t)^3 \\
3h^2R^2x & = & r^2(x+t)^3 & &
\end{array}$$

Differentiating implicitly with respect to t gives us $3h^2R^2\,\dfrac{dx}{dt} = r^2\left[3(x+t)^2\,\dfrac{dx}{dt} + 3(x+t)^2\,\dfrac{dt}{dt}\right] \;\Rightarrow$

$\dfrac{dx}{dt} = \dfrac{r^2(x+t)^2}{h^2R^2 - r^2(x+t)^2} \;\Rightarrow\; \left.\dfrac{dx}{dt}\right|_{x+t=h} = \dfrac{r^2h^2}{h^2R^2 - r^2h^2} = \dfrac{r^2}{R^2 - r^2}$. Thus, the water level is rising at a rate of $\dfrac{r^2}{R^2 - r^2}$ cm/s at the instant the cone is completely submerged.

31.



By similar triangles, $\dfrac{r}{5} = \dfrac{h}{16} \;\Rightarrow\; r = \dfrac{5h}{16}$. The volume of the cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\dfrac{5h}{16}\right)^2 h = \dfrac{25\pi}{768}h^3$, so $\dfrac{dV}{dt} = \dfrac{25\pi}{256}h^2\,\dfrac{dh}{dt}$. Now the rate of change of the volume is also equal to the difference of what is being added ($2\text{ cm}^3/\text{min}$) and what is oozing out ($k\pi rl$, where πrl is the area of the cone and k is a proportionality constant). Thus, $\dfrac{dV}{dt} = 2 - k\pi rl$.

Equating the two expressions for $\dfrac{dV}{dt}$ and substituting $h = 10$, $\dfrac{dh}{dt} = -0.3$, $r = \dfrac{5(10)}{16} = \dfrac{25}{8}$, and $\dfrac{l}{\sqrt{281}} = \dfrac{10}{16} \;\Leftrightarrow$ $l = \frac{5}{8}\sqrt{281}$, we get $\dfrac{25\pi}{256}(10)^2(-0.3) = 2 - k\pi\dfrac{25}{8}\cdot\dfrac{5}{8}\sqrt{281} \;\Leftrightarrow\; \dfrac{125k\pi\sqrt{281}}{64} = 2 + \dfrac{750\pi}{256}$. Solving for k gives us $k = \dfrac{256 + 375\pi}{250\pi\sqrt{281}}$. To maintain a certain height, the rate of oozing, $k\pi rl$, must equal the rate of the liquid being poured in; that is, $\dfrac{dV}{dt} = 0$. Thus, the rate at which we should pour the liquid into the container is

$$k\pi rl = \frac{256 + 375\pi}{250\pi\sqrt{281}}\cdot\pi\cdot\frac{25}{8}\cdot\frac{5\sqrt{281}}{8} = \frac{256 + 375\pi}{128} \approx 11.204\text{ cm}^3/\text{min}$$