4.4 Indeterminate Forms and L'Hospital's Rule

Note: The use of l'Hospital's Rule is indicated by an H above the equal sign: $\overset{\text{H}}{=}$

1. (a) $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ is an indeterminate form of type $\dfrac{0}{0}$.

   (b) $\lim_{x \to a} \dfrac{f(x)}{p(x)} = 0$ because the numerator approaches 0 while the denominator becomes large.

   (c) $\lim_{x \to a} \dfrac{h(x)}{p(x)} = 0$ because the numerator approaches a finite number while the denominator becomes large.

   (d) If $\lim_{x \to a} p(x) = \infty$ and $f(x) \to 0$ through positive values, then $\lim_{x \to a} \dfrac{p(x)}{f(x)} = \infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x^2$.] If $f(x) \to 0$ through negative values, then $\lim_{x \to a} \dfrac{p(x)}{f(x)} = -\infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = -x^2$.] If $f(x) \to 0$ through both positive and negative values, then the limit might not exist. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x$.]

   (e) $\lim_{x \to a} \dfrac{p(x)}{q(x)}$ is an indeterminate form of type $\dfrac{\infty}{\infty}$.

2. (a) $\lim_{x \to a} [f(x)p(x)]$ is an indeterminate form of type $0 \cdot \infty$.

   (b) When x is near a, $p(x)$ is large and $h(x)$ is near 1, so $h(x)p(x)$ is large. Thus, $\lim_{x \to a} [h(x)p(x)] = \infty$.

   (c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x)q(x)$ is large. Thus, $\lim_{x \to a} [p(x)q(x)] = \infty$.

3. (a) When x is near a, $f(x)$ is near 0 and $p(x)$ is large, so $f(x) - p(x)$ is large negative. Thus, $\lim_{x \to a} [f(x) - p(x)] = -\infty$.

   (b) $\lim_{x \to a} [p(x) - q(x)]$ is an indeterminate form of type $\infty - \infty$.

   (c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x) + q(x)$ is large. Thus, $\lim_{x \to a} [p(x) + q(x)] = \infty$.

4. (a) $\lim_{x \to a} [f(x)]^{g(x)}$ is an indeterminate form of type 0^0.

   (b) If $y = [f(x)]^{p(x)}$, then $\ln y = p(x) \ln f(x)$. When x is near a, $p(x) \to \infty$ and $\ln f(x) \to -\infty$, so $\ln y \to -\infty$. Therefore, $\lim_{x \to a} [f(x)]^{p(x)} = \lim_{x \to a} y = \lim_{x \to a} e^{\ln y} = 0$, provided f^p is defined.

   (c) $\lim_{x \to a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞.

   (d) $\lim_{x \to a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0.

   (e) If $y = [p(x)]^{q(x)}$, then $\ln y = q(x) \ln p(x)$. When x is near a, $q(x) \to \infty$ and $\ln p(x) \to \infty$, so $\ln y \to \infty$. Therefore, $\lim_{x \to a} [p(x)]^{q(x)} = \lim_{x \to a} y = \lim_{x \to a} e^{\ln y} = \infty$.

   (f) $\lim_{x \to a} \sqrt[q(x)]{p(x)} = \lim_{x \to a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0.

5. This limit has the form $\frac{0}{0}$. We can simply factor and simplify to evaluate the limit.

$$\lim_{x\to 1}\frac{x^2-1}{x^2-x}=\lim_{x\to 1}\frac{(x+1)(x-1)}{x(x-1)}=\lim_{x\to 1}\frac{x+1}{x}=\frac{1+1}{1}=2$$

6. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 2}\dfrac{x^2+x-6}{x-2}=\lim\limits_{x\to 2}\dfrac{(x+3)(x-2)}{x-2}=\lim\limits_{x\to 2}(x+3)=2+3=5$

7. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1}\dfrac{x^9-1}{x^5-1}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{9x^8}{5x^4}=\dfrac{9}{5}\lim\limits_{x\to 1}x^4=\dfrac{9}{5}(1)=\dfrac{9}{5}$

8. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1}\dfrac{x^a-1}{x^b-1}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{ax^{a-1}}{bx^{b-1}}=\dfrac{a}{b}$

9. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to(\pi/2)^+}\dfrac{\cos x}{1-\sin x}\overset{\text{H}}{=}\lim\limits_{x\to(\pi/2)^+}\dfrac{-\sin x}{-\cos x}=\lim\limits_{x\to(\pi/2)^+}\tan x=-\infty.$

10. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{\sin 4x}{\tan 5x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{4\cos 4x}{5\sec^2(5x)}=\dfrac{4(1)}{5(1)^2}=\dfrac{4}{5}$

11. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to 0}\dfrac{e^t-1}{t^3}\overset{\text{H}}{=}\lim\limits_{t\to 0}\dfrac{e^t}{3t^2}=\infty$ since $e^t\to 1$ and $3t^2\to 0^+$ as $t\to 0$.

12. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to 0}\dfrac{e^{3t}-1}{t}\overset{\text{H}}{=}\lim\limits_{t\to 0}\dfrac{3e^{3t}}{1}=3$

13. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{\tan px}{\tan qx}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{p\sec^2 px}{q\sec^2 qx}=\dfrac{p(1)^2}{q(1)^2}=\dfrac{p}{q}$

14. $\lim\limits_{\theta\to\pi/2}\dfrac{1-\sin\theta}{\csc\theta}=\dfrac{0}{1}=0.$ L'Hospital's Rule does not apply.

15. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{\ln x}{\sqrt{x}}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1/x}{\frac{1}{2}x^{-1/2}}=\lim\limits_{x\to\infty}\dfrac{2}{\sqrt{x}}=0$

16. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{x+x^2}{1-2x^2}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1+2x}{-4x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{2}{-4}=-\dfrac{1}{2}.$

A better method is to divide the numerator and the denominator by x^2: $\lim\limits_{x\to\infty}\dfrac{x+x^2}{1-2x^2}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}+1}{\frac{1}{x^2}-2}=\dfrac{0+1}{0-2}=-\dfrac{1}{2}.$

17. $\lim\limits_{x\to 0^+}[(\ln x)/x]=-\infty$ since $\ln x\to-\infty$ as $x\to 0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.

18. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{\ln\ln x}{x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1}=\lim\limits_{x\to\infty}\dfrac{1}{x\ln x}=0$

19. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{e^x}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{3x^2}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{6x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{6}=\infty$

20. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1}\dfrac{\ln x}{\sin\pi x}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{1/x}{\pi\cos\pi x}=\dfrac{1}{\pi(-1)}=-\dfrac{1}{\pi}$

21. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{e^x-1-x}{x^2}\overset{\text{H}}{=}\lim_{x\to 0}\frac{e^x-1}{2x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{e^x}{2}=\frac{1}{2}$

22. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{e^x-1-x-\frac{1}{2}x^2}{x^3}\overset{\text{H}}{=}\lim_{x\to 0}\frac{e^x-1-x}{3x^2}\overset{\text{H}}{=}\lim_{x\to 0}\frac{e^x-1}{6x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{e^x}{6}=\frac{1}{6}$

23. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{\tanh x}{\tan x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\text{sech}^2 x}{\sec^2 x}=\frac{\text{sech}^2\, 0}{\sec^2 0}=\frac{1}{1}=1$

24. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{1-\cos x}{1-\sec^2 x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-(-\sin x)}{-2\sec x\,(\sec x\,\tan x)}=-\frac{1}{2}\lim_{x\to 0}\frac{\sin x\left(\dfrac{\cos x}{\sin x}\right)}{\sec^2 x}$$

$$=-\tfrac{1}{2}\lim_{x\to 0}\cos^3 x=-\tfrac{1}{2}(1)^3=-\tfrac{1}{2}$$

Another method is to write the limit as $\displaystyle\lim_{x\to 0}\frac{1-\dfrac{\sin x}{x}}{1-\dfrac{\tan x}{x}}$.

25. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{t\to 0}\frac{5^t-3^t}{t}\overset{\text{H}}{=}\lim_{t\to 0}\frac{5^t\ln 5-3^t\ln 3}{1}=\ln 5-\ln 3=\ln\tfrac{5}{3}$

26. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{\sin x-x}{x^3}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\cos x-1}{3x^2}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-\sin x}{6x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-\cos x}{6}=-\frac{1}{6}$

27. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x}{x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{1/\sqrt{1-x^2}}{1}=\lim_{x\to 0}\frac{1}{\sqrt{1-x^2}}=\frac{1}{1}=1$

28. This limit has the form $\frac{\infty}{\infty}$. $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^2}{x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{2(\ln x)(1/x)}{1}=2\lim_{x\to\infty}\frac{\ln x}{x}\overset{\text{H}}{=}2\lim_{x\to\infty}\frac{1/x}{1}=2(0)=0$

29. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\sin x}{2x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$

30. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-m\sin mx+n\sin nx}{2x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-m^2\cos mx+n^2\cos nx}{2}=\tfrac{1}{2}\left(n^2-m^2\right)$$

31. $\displaystyle\lim_{x\to 0}\frac{x+\sin x}{x+\cos x}=\frac{0+0}{0+1}=\frac{0}{1}=0$. L'Hospital's Rule does not apply.

32. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{x}{\tan^{-1}(4x)}\overset{\text{H}}{=}\lim_{x\to 0}\frac{1}{\dfrac{1}{1+(4x)^2}\cdot 4}=\lim_{x\to 0}\frac{1+16x^2}{4}=\frac{1}{4}$

33. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 1}\frac{1-x+\ln x}{1+\cos\pi x}\overset{\text{H}}{=}\lim_{x\to 1}\frac{-1+1/x}{-\pi\sin\pi x}\overset{\text{H}}{=}\lim_{x\to 1}\frac{-1/x^2}{-\pi^2\cos\pi x}=\frac{-1}{-\pi^2(-1)}=-\frac{1}{\pi^2}$

34. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x^2+2}}{\sqrt{2x^2+1}}=\lim_{x\to\infty}\sqrt{\frac{x^2+2}{2x^2+1}}=\sqrt{\lim_{x\to\infty}\frac{x^2+2}{2x^2+1}}=\sqrt{\lim_{x\to\infty}\frac{1+2/x^2}{2+1/x^2}}=\sqrt{\frac{1}{2}}$

35. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 1}\frac{x^a-ax+a-1}{(x-1)^2}\overset{\text{H}}{=}\lim_{x\to 1}\frac{ax^{a-1}-a}{2(x-1)}\overset{\text{H}}{=}\lim_{x\to 1}\frac{a(a-1)x^{a-2}}{2}=\frac{a(a-1)}{2}$

36. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{e^x + e^{-x} - 2}{1 - \cos x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{\sin x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{e^x + e^{-x}}{\cos x} = \dfrac{1+1}{1} = 2$

37. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0} \dfrac{\cos x - 1 + \frac{1}{2}x^2}{x^4} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{-\sin x + x}{4x^3} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{-\cos x + 1}{12x^2} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{\sin x}{24x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{\cos x}{24} = \dfrac{1}{24}$

38. This limit has the form $\frac{\infty}{\infty}$.

$$\lim_{x\to a^+} \frac{\cos x \ln(x-a)}{\ln(e^x - e^a)} = \lim_{x\to a^+} \cos x \lim_{x\to a^+} \frac{\ln(x-a)}{\ln(e^x-e^a)} \overset{\text{H}}{=} \cos a \lim_{x\to a^+} \frac{\frac{1}{x-a}}{\frac{1}{e^x - e^a}\cdot e^x}$$

$$= \cos a \lim_{x\to a^+} \frac{1}{e^x} \cdot \lim_{x\to a^+} \frac{e^x - e^a}{x-a} \overset{\text{H}}{=} \cos a \cdot \frac{1}{e^a} \lim_{x\to a^+} \frac{e^x}{1} = \cos a \cdot \frac{1}{e^a} \cdot e^a = \cos a$$

39. This limit has the form $\infty \cdot 0$.

$$\lim_{x\to\infty} x \sin(\pi/x) = \lim_{x\to\infty} \frac{\sin(\pi/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\cos(\pi/x)(-\pi/x^2)}{-1/x^2} = \pi \lim_{x\to\infty} \cos(\pi/x) = \pi(1) = \pi$$

40. This limit has the form $\infty \cdot 0$. $\lim\limits_{x\to-\infty} x^2 e^x = \lim\limits_{x\to-\infty} \dfrac{x^2}{e^{-x}} \overset{\text{H}}{=} \lim\limits_{x\to-\infty} \dfrac{2x}{-e^{-x}} \overset{\text{H}}{=} \lim\limits_{x\to-\infty} \dfrac{2}{e^{-x}} = \lim\limits_{x\to-\infty} 2e^x = 0$

41. This limit has the form $\infty \cdot 0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x\to 0} \cot 2x \sin 6x = \lim_{x\to 0} \frac{\sin 6x}{\tan 2x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{6\cos 6x}{2\sec^2 2x} = \frac{6(1)}{2(1)^2} = 3$$

42. This limit has the form $0 \cdot (-\infty)$.

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\csc x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} = -\lim_{x\to 0^+} \left(\frac{\sin x}{x} \cdot \tan x\right) = -\left(\lim_{x\to 0^+} \frac{\sin x}{x}\right)\left(\lim_{x\to 0^+} \tan x\right)$$

$$= -1 \cdot 0 = 0$$

43. This limit has the form $\infty \cdot 0$. $\lim\limits_{x\to\infty} x^3 e^{-x^2} = \lim\limits_{x\to\infty} \dfrac{x^3}{e^{x^2}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{3x^2}{2xe^{x^2}} = \lim\limits_{x\to\infty} \dfrac{3x}{2e^{x^2}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{3}{4xe^{x^2}} = 0$

44. $\lim\limits_{x\to\pi/4} (1 - \tan x)\sec x = (1-1)\sqrt{2} = 0$. L'Hospital's Rule does not apply.

45. This limit has the form $0 \cdot (-\infty)$.

$$\lim_{x\to 1^+} \ln x \, \tan(\pi x/2) = \lim_{x\to 1^+} \frac{\ln x}{\cot(\pi x/2)} \overset{\text{H}}{=} \lim_{x\to 1^+} \frac{1/x}{(-\pi/2)\csc^2(\pi x/2)} = \frac{1}{(-\pi/2)(1)^2} = -\frac{2}{\pi}$$

46. This limit has the form $\infty \cdot 0$.

$$\lim_{x\to\infty} x \tan(1/x) = \lim_{x\to\infty} \frac{\tan(1/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\sec^2(1/x)(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \sec^2(1/x) = 1^2 = 1$$

47. This limit has the form $\infty - \infty$.

$$\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1} \frac{x\ln x - (x-1)}{(x-1)\ln x} \overset{\text{H}}{=} \lim_{x\to 1} \frac{x(1/x) + \ln x - 1}{(x-1)(1/x) + \ln x} = \lim_{x\to 1} \frac{\ln x}{1 - (1/x) + \ln x}$$

$$\overset{\text{H}}{=} \lim_{x\to 1} \frac{1/x}{1/x^2 + 1/x} \cdot \frac{x^2}{x^2} = \lim_{x\to 1} \frac{x}{1+x} = \frac{1}{1+1} = \frac{1}{2}$$

48. This limit has the form $\infty - \infty$. $\lim\limits_{x\to 0} (\csc x - \cot x) = \lim\limits_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}\right) = \lim\limits_{x\to 0} \dfrac{1 - \cos x}{\sin x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{\sin x}{\cos x} = 0$

49. We will multiply and divide by the conjugate of the expression to change the form of the expression.

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to\infty}\left(\frac{\sqrt{x^2+x}-x}{1}\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right)=\lim_{x\to\infty}\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}$$
$$=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{2}$$

As an alternate solution, write $\sqrt{x^2+x}-x$ as $\sqrt{x^2+x}-\sqrt{x^2}$, factor out $\sqrt{x^2}$, rewrite as $\left(\sqrt{1+1/x}-1\right)/(1/x)$, and apply l'Hospital's Rule.

50. This limit has the form $\infty-\infty$.

$$\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{\cos x}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin x}\overset{H}{=}\lim_{x\to 0}\frac{x(-\sin x)+\cos x-\cos x}{x\cos x+\sin x}$$
$$=-\lim_{x\to 0}\frac{x\sin x}{x\cos x+\sin x}\overset{H}{=}-\lim_{x\to 0}\frac{x\cos x+\sin x}{x(-\sin x)+\cos x+\cos x}=-\frac{0+0}{0+1+1}=0$$

51. The limit has the form $\infty-\infty$ and we will change the form to a product by factoring out x.

$$\lim_{x\to\infty}(x-\ln x)=\lim_{x\to\infty}x\left(1-\frac{\ln x}{x}\right)=\infty \text{ since } \lim_{x\to\infty}\frac{\ln x}{x}\overset{H}{=}\lim_{x\to\infty}\frac{1/x}{1}=0.$$

52. As $x\to\infty$, $1/x\to 0$, and $e^{1/x}\to 1$. So the limit has the form $\infty-\infty$ and we will change the form to a product by factoring out x.

$$\lim_{x\to\infty}\left(xe^{1/x}-x\right)=\lim_{x\to\infty}x\left(e^{1/x}-1\right)=\lim_{x\to\infty}\frac{e^{1/x}-1}{1/x}\overset{H}{=}\lim_{x\to\infty}\frac{e^{1/x}(-1/x^2)}{-1/x^2}=\lim_{x\to\infty}e^{1/x}=e^0=1$$

53. $y=x^{x^2}\quad\Rightarrow\quad \ln y=x^2\ln x$, so $\displaystyle\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}x^2\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/x^2}\overset{H}{=}\lim_{x\to 0^+}\frac{1/x}{-2/x^3}=\lim_{x\to 0^+}\left(-\frac{1}{2}x^2\right)=0\quad\Rightarrow$

$$\lim_{x\to 0^+}x^{x^2}=\lim_{x\to 0^+}e^{\ln y}=e^0=1.$$

54. $y=(\tan 2x)^x\quad\Rightarrow\quad \ln y=x\cdot\ln\tan 2x$, so

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}x\cdot\ln\tan 2x=\lim_{x\to 0^+}\frac{\ln\tan 2x}{1/x}\overset{H}{=}\lim_{x\to 0^+}\frac{(1/\tan 2x)(2\sec^2 2x)}{-1/x^2}=\lim_{x\to 0^+}\frac{-2x^2\cos 2x}{\sin 2x\cos^2 2x}$$
$$=\lim_{x\to 0^+}\frac{2x}{\sin 2x}\cdot\lim_{x\to 0^+}\frac{-x}{\cos 2x}=1\cdot 0=0\quad\Rightarrow$$
$$\lim_{x\to 0^+}(\tan 2x)^x=\lim_{x\to 0^+}e^{\ln y}=e^0=1.$$

55. $y=(1-2x)^{1/x}\quad\Rightarrow\quad \ln y=\dfrac{1}{x}\ln(1-2x)$, so $\displaystyle\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(1-2x)}{x}\overset{H}{=}\lim_{x\to 0}\frac{-2/(1-2x)}{1}=-2\quad\Rightarrow$

$$\lim_{x\to 0}(1-2x)^{1/x}=\lim_{x\to 0}e^{\ln y}=e^{-2}.$$

56. $y=\left(1+\dfrac{a}{x}\right)^{bx}\quad\Rightarrow\quad \ln y=bx\ln\left(1+\dfrac{a}{x}\right)$, so

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{b\ln(1+a/x)}{1/x}\overset{H}{=}\lim_{x\to\infty}\frac{b\left(\dfrac{1}{1+a/x}\right)\left(-\dfrac{a}{x^2}\right)}{-1/x^2}=\lim_{x\to\infty}\frac{ab}{1+a/x}=ab\quad\Rightarrow$$
$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=\lim_{x\to\infty}e^{\ln y}=e^{ab}.$$

57. $y = \left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)^x \Rightarrow \ln y = x \ln\left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right) \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\left(-\dfrac{3}{x^2} - \dfrac{10}{x^3}\right)\Big/\left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{3 + \dfrac{10}{x}}{1 + \dfrac{3}{x} + \dfrac{5}{x^2}} = 3,$$

so $\displaystyle\lim_{x\to\infty}\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^3.$

58. $y = x^{(\ln 2)/(1 + \ln x)} \Rightarrow \ln y = \dfrac{\ln 2}{1 + \ln x} \ln x \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{(\ln 2)(\ln x)}{1 + \ln x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{(\ln 2)(1/x)}{1/x} = \lim_{x\to\infty} \ln 2 = \ln 2, \text{ so } \lim_{x\to\infty} x^{(\ln 2)/(1 + \ln x)} = \lim_{x\to\infty} e^{\ln y} = e^{\ln 2} = 2.$$

59. $y = x^{1/x} \Rightarrow \ln y = (1/x)\ln x \Rightarrow \displaystyle\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0 \Rightarrow$

$\displaystyle\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$

60. $y = (e^x + x)^{1/x} \Rightarrow \ln y = \dfrac{1}{x}\ln(e^x + x),$

so $\displaystyle\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(e^x + x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x + 1}{e^x + x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x + 1} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = 1 \Rightarrow$

$\displaystyle\lim_{x\to\infty} (e^x + x)^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^1 = e.$

61. $y = (4x + 1)^{\cot x} \Rightarrow \ln y = \cot x \ln(4x + 1)$, so $\displaystyle\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(4x + 1)}{\tan x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\dfrac{4}{4x + 1}}{\sec^2 x} = 4 \Rightarrow$

$\displaystyle\lim_{x\to 0^+} (4x + 1)^{\cot x} = \lim_{x\to 0^+} e^{\ln y} = e^4.$

62. $y = (2 - x)^{\tan(\pi x/2)} \Rightarrow \ln y = \tan\left(\dfrac{\pi x}{2}\right)\ln(2 - x) \Rightarrow$

$$\lim_{x\to 1} \ln y = \lim_{x\to 1}\left[\tan\left(\frac{\pi x}{2}\right)\ln(2 - x)\right] = \lim_{x\to 1} \frac{\ln(2 - x)}{\cot\left(\dfrac{\pi x}{2}\right)} \overset{\text{H}}{=} \lim_{x\to 1} \frac{\dfrac{1}{2 - x}(-1)}{-\csc^2\left(\dfrac{\pi x}{2}\right)\cdot\dfrac{\pi}{2}} = \frac{2}{\pi}\lim_{x\to 1} \frac{\sin^2\left(\dfrac{\pi x}{2}\right)}{2 - x}$$

$$= \frac{2}{\pi}\cdot\frac{1^2}{1} = \frac{2}{\pi} \Rightarrow \lim_{x\to 1}(2 - x)^{\tan(\pi x/2)} = \lim_{x\to 1} e^{\ln y} = e^{(2/\pi)}$$

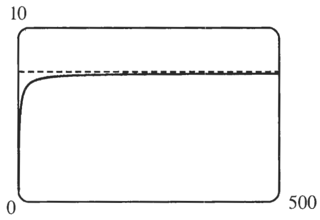
63. $y = (\cos x)^{1/x^2} \Rightarrow \ln y = \dfrac{1}{x^2}\ln\cos x \Rightarrow \displaystyle\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln\cos x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{-\tan x}{2x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{-\sec^2 x}{2} = -\frac{1}{2}$

$\Rightarrow \displaystyle\lim_{x\to 0^+} (\cos x)^{1/x^2} = \lim_{x\to 0^+} e^{\ln y} = e^{-1/2} = 1/\sqrt{e}$

64. $y = \left(\dfrac{2x - 3}{2x + 5}\right)^{2x+1} \Rightarrow \ln y = (2x + 1)\ln\left(\dfrac{2x - 3}{2x + 5}\right) \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(2x - 3) - \ln(2x + 5)}{1/(2x + 1)} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{2/(2x - 3) - 2/(2x + 5)}{-2/(2x + 1)^2} = \lim_{x\to\infty} \frac{-8(2x + 1)^2}{(2x - 3)(2x + 5)}$$

$$= \lim_{x\to\infty} \frac{-8(2 + 1/x)^2}{(2 - 3/x)(2 + 5/x)} = -8 \Rightarrow \lim_{x\to\infty}\left(\frac{2x - 3}{2x + 5}\right)^{2x+1} = e^{-8}$$

65. 

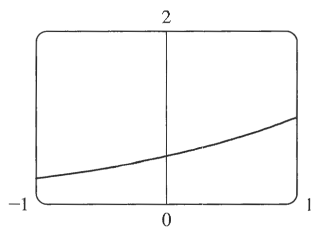
From the graph, if $x = 500$, $y \approx 7.36$. The limit has the form 1^∞.

Now $y = \left(1 + \frac{2}{x}\right)^x \Rightarrow \ln y = x \ln\left(1 + \frac{2}{x}\right) \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+2/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+2/x}\left(-\frac{2}{x^2}\right)}{-1/x^2}$$

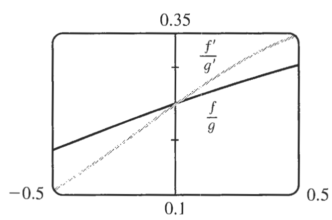
$$= 2 \lim_{x\to\infty} \frac{1}{1+2/x} = 2(1) = 2 \Rightarrow$$

$$\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^2 \; [\approx 7.39]$$

66. 

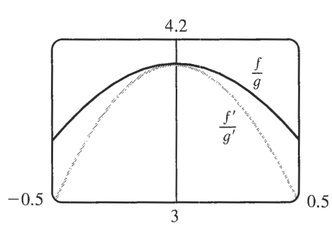
From the graph, as $x \to 0$, $y \approx 0.55$. The limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{5^x - 4^x}{3^x - 2^x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{5^x \ln 5 - 4^x \ln 4}{3^x \ln 3 - 2^x \ln 2} = \frac{\ln 5 - \ln 4}{\ln 3 - \ln 2} = \frac{\ln \frac{5}{4}}{\ln \frac{3}{2}} \; [\approx 0.55]$$

67. 

From the graph, it appears that $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = 0.25$.

We calculate $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{e^x - 1}{x^3 + 4x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{e^x}{3x^2 + 4} = \frac{1}{4}$.

68. 

From the graph, it appears that $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = 4$. We calculate

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{2x \sin x}{\sec x - 1} \overset{\text{H}}{=} \lim_{x\to 0} \frac{2(x\cos x + \sin x)}{\sec x \, \tan x}$$

$$\overset{\text{H}}{=} \lim_{x\to 0} \frac{2(-x\sin x + \cos x + \cos x)}{\sec x(\sec^2 x) + \tan x(\sec x \, \tan x)} = \frac{4}{1} = 4$$

69. $\lim_{x\to\infty} \frac{e^x}{x^n} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{n!} = \infty$

70. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty} \frac{\ln x}{x^p} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0$ since $p > 0$.

71. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{2}(x^2+1)^{-1/2}(2x)} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$. Repeated applications of l'Hospital's Rule result in the original limit or the limit of the reciprocal of the function. Another method is to try dividing the numerator and denominator by x: $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2/x^2 + 1/x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^2}} = \frac{1}{1} = 1$

72. (a) $\lim_{t\to\infty} v = \lim_{t\to\infty} \frac{mg}{c}\left(1 - e^{-ct/m}\right) = \frac{mg}{c} \lim_{t\to\infty}\left(1 - e^{-ct/m}\right) = \frac{mg}{c}(1 - 0)$ [because $-ct/m \to -\infty$ as $t \to \infty$]

$= \frac{mg}{c}$, which is the speed the object approaches as time goes on, the so-called limiting velocity.

(b) $\lim\limits_{c\to 0^+} v = \lim\limits_{c\to 0^+} \frac{mg}{c}(1 - e^{-ct/m}) = mg \lim\limits_{c\to 0^+} \frac{1 - e^{-ct/m}}{c}$ [form is $\frac{0}{0}$]

$$\overset{\text{H}}{=} mg \lim_{c\to 0^+} \frac{(-e^{-ct/m})\cdot(-t/m)}{1} = \frac{mgt}{m} \lim_{c\to 0^+} e^{-ct/m} = gt(1) = gt$$

The velocity of a falling object in a vacuum is directly proportional to the amount of time it falls.

73. First we will find $\lim\limits_{n\to\infty}\left(1 + \frac{r}{n}\right)^{nt}$, which is of the form 1^∞. $y = \left(1 + \frac{r}{n}\right)^{nt} \Rightarrow \ln y = nt\ln\left(1 + \frac{r}{n}\right)$, so

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} nt\ln\left(1 + \frac{r}{n}\right) = t \lim_{n\to\infty} \frac{\ln(1 + r/n)}{1/n} \overset{\text{H}}{=} t \lim_{n\to\infty} \frac{(-r/n^2)}{(1 + r/n)(-1/n^2)} = t \lim_{n\to\infty} \frac{r}{1 + i/n} = tr \Rightarrow$$

$\lim\limits_{n\to\infty} y = e^{rt}$. Thus, as $n \to \infty$, $A = A_0\left(1 + \frac{r}{n}\right)^{nt} \to A_0 e^{rt}$.

74. $\lim\limits_{c\to 0^+} s(t) = \lim\limits_{c\to 0^+}\left(\frac{m}{c}\ln\cosh\sqrt{\frac{gc}{mt}}\right) = m \lim\limits_{c\to 0^+} \frac{\ln\cosh\sqrt{ac}}{c}$ [let $a = g/(mt)$]

$$\overset{\text{H}}{=} m \lim_{c\to 0^+} \frac{\frac{1}{\cosh\sqrt{ac}}(\sinh\sqrt{ac})\left(\frac{\sqrt{a}}{2\sqrt{c}}\right)}{1} = \frac{m\sqrt{a}}{2} \lim_{c\to 0^+} \frac{\tanh\sqrt{ac}}{\sqrt{c}}$$

$$\overset{\text{H}}{=} \frac{m\sqrt{a}}{2} \lim_{c\to 0^+} \frac{\operatorname{sech}^2\sqrt{ac}\left[\sqrt{a}/\left(2\sqrt{c}\right)\right]}{1/\left(2\sqrt{c}\right)} = \frac{ma}{2} \lim_{c\to 0^+} \operatorname{sech}^2\sqrt{ac} = \frac{ma}{2}(1)^2 = \frac{mg}{2mt} = \frac{g}{2t}$$

75. $\lim\limits_{E\to 0^+} P(E) = \lim\limits_{E\to 0^+}\left(\frac{e^E + e^{-E}}{e^E - e^{-E}} - \frac{1}{E}\right)$

$$= \lim_{E\to 0^+} \frac{E(e^E + e^{-E}) - 1(e^E - e^{-E})}{(e^E - e^{-E})E} = \lim_{E\to 0^+} \frac{Ee^E + Ee^{-E} - e^E + e^{-E}}{Ee^E - Ee^{-E}} \quad \left[\text{form is } \tfrac{0}{0}\right]$$

$$\overset{\text{H}}{=} \lim_{E\to 0^+} \frac{Ee^E + e^E\cdot 1 + E(-e^{-E}) + e^{-E}\cdot 1 - e^E + (-e^{-E})}{Ee^E + e^E\cdot 1 - [E(-e^{-E}) + e^{-E}\cdot 1]}$$

$$= \lim_{E\to 0^+} \frac{Ee^E - Ee^{-E}}{Ee^E + e^E + Ee^{-E} - e^{-E}} = \lim_{E\to 0^+} \frac{e^E - e^{-E}}{e^E + \frac{e^E}{E} + e^{-E} - \frac{e^{-E}}{E}} \quad [\text{divide by } E]$$

$$= \frac{0}{2 + L}, \text{ where } L = \lim_{E\to 0^+} \frac{e^E - e^{-E}}{E} \quad \left[\text{form is } \tfrac{0}{0}\right] \overset{\text{H}}{=} \lim_{E\to 0^+} \frac{e^E + e^{-E}}{1} = \frac{1 + 1}{1} = 2$$

Thus, $\lim\limits_{E\to 0^+} P(E) = \frac{0}{2 + 2} = 0$.

76. (a) $\lim\limits_{R\to r^+} v = \lim\limits_{R\to r^+}\left[-c\left(\frac{r}{R}\right)^2\ln\left(\frac{r}{R}\right)\right] = -cr^2 \lim\limits_{R\to r^+}\left[\left(\frac{1}{R}\right)^2\ln\left(\frac{r}{R}\right)\right] = -cr^2\cdot\frac{1}{r^2}\cdot\ln 1 = -c\cdot 0 = 0$

As the insulation of a metal cable becomes thinner, the velocity of an electrical impulse in the cable approaches zero.

(b) $\lim\limits_{r\to 0^+} v = \lim\limits_{r\to 0^+}\left[-c\left(\frac{r}{R}\right)^2\ln\left(\frac{r}{R}\right)\right] = -\frac{c}{R^2}\lim\limits_{r\to 0^+}\left[r^2\ln\left(\frac{r}{R}\right)\right]$ [form is $0\cdot\infty$]

$$= -\frac{c}{R^2}\lim_{r\to 0^+}\frac{\ln\left(\frac{r}{R}\right)}{\frac{1}{r^2}} \quad [\text{form is } \infty/\infty] \quad \overset{\text{H}}{=} -\frac{c}{R^2}\lim_{r\to 0^+}\frac{\frac{R}{r}\cdot\frac{1}{R}}{\frac{-2}{r^3}} = -\frac{c}{R^2}\lim_{r\to 0^+}\left(-\frac{r^2}{2}\right) = 0$$

As the radius of the metal cable approaches zero, the velocity of an electrical impulse in the cable approaches zero.

77. We see that both numerator and denominator approach 0, so we can use l'Hospital's Rule:

$$\begin{aligned}\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{aax}}{a-\sqrt[4]{ax^3}} &\overset{\text{H}}{=} \lim_{x\to a}\frac{\frac12(2a^3x-x^4)^{-1/2}(2a^3-4x^3)-a\left(\frac13\right)(aax)^{-2/3}a^2}{-\frac14(ax^3)^{-3/4}(3ax^2)}\\ &= \frac{\frac12(2a^3a-a^4)^{-1/2}(2a^3-4a^3)-\frac13a^3(a^2a)^{-2/3}}{-\frac14(aa^3)^{-3/4}(3aa^2)}\\ &= \frac{(a^4)^{-1/2}(-a^3)-\frac13a^3(a^3)^{-2/3}}{-\frac34a^3(a^4)^{-3/4}} = \frac{-a-\frac13a}{-\frac34} = \frac43\left(\frac43a\right) = \frac{16}{9}a\end{aligned}$$

78. Let the radius of the circle be r. We see that $A(\theta)$ is the area of the whole figure (a sector of the circle with radius 1), minus the area of $\triangle OPR$. But the area of the sector of the circle is $\frac12r^2\theta$ (see Reference Page 1), and the area of the triangle is $\frac12r\,|PQ| = \frac12r(r\sin\theta) = \frac12r^2\sin\theta$. So we have $A(\theta) = \frac12r^2\theta - \frac12r^2\sin\theta = \frac12r^2(\theta-\sin\theta)$. Now by elementary trigonometry, $B(\theta) = \frac12\,|QR|\,|PQ| = \frac12(r-|OQ|)\,|PQ| = \frac12(r-r\cos\theta)(r\sin\theta) = \frac12r^2(1-\cos\theta)\sin\theta$.

So the limit we want is

$$\begin{aligned}\lim_{\theta\to0^+}\frac{A(\theta)}{B(\theta)} &= \lim_{\theta\to0^+}\frac{\frac12r^2(\theta-\sin\theta)}{\frac12r^2(1-\cos\theta)\sin\theta} \overset{\text{H}}{=} \lim_{\theta\to0^+}\frac{1-\cos\theta}{(1-\cos\theta)\cos\theta+\sin\theta\,(\sin\theta)}\\ &= \lim_{\theta\to0^+}\frac{1-\cos\theta}{\cos\theta-\cos^2\theta+\sin^2\theta} \overset{\text{H}}{=} \lim_{\theta\to0^+}\frac{\sin\theta}{-\sin\theta-2\cos\theta\,(-\sin\theta)+2\sin\theta\,(\cos\theta)}\\ &= \lim_{\theta\to0^+}\frac{\sin\theta}{-\sin\theta+4\sin\theta\,\cos\theta} = \lim_{\theta\to0^+}\frac{1}{-1+4\cos\theta} = \frac{1}{-1+4\cos0} = \frac13\end{aligned}$$

79. Since $f(2)=0$, the given limit has the form $\frac00$.

$$\lim_{x\to0}\frac{f(2+3x)+f(2+5x)}{x} \overset{\text{H}}{=} \lim_{x\to0}\frac{f'(2+3x)\cdot3+f'(2+5x)\cdot5}{1} = f'(2)\cdot3+f'(2)\cdot5 = 8f'(2) = 8\cdot7 = 56$$

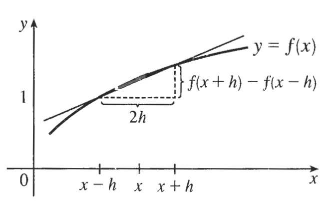
80. $L = \lim_{x\to0}\left(\frac{\sin2x}{x^3}+a+\frac{b}{x^2}\right) = \lim_{x\to0}\frac{\sin2x+ax^3+bx}{x^3} \overset{\text{H}}{=} \lim_{x\to0}\frac{2\cos2x+3ax^2+b}{3x^2}$. As $x\to0$, $3x^2\to0$, and $\left(2\cos2x+3ax^2+b\right)\to b+2$, so the last limit exists only if $b+2=0$, that is, $b=-2$. Thus,

$\lim_{x\to0}\frac{2\cos2x+3ax^2-2}{3x^2} \overset{\text{H}}{=} \lim_{x\to0}\frac{-4\sin2x+6ax}{6x} \overset{\text{H}}{=} \lim_{x\to0}\frac{-8\cos2x+6a}{6} = \frac{6a-8}{6}$, which is equal to 0 if and only if $a=\frac43$. Hence, $L=0$ if and only if $b=-2$ and $a=\frac43$.

81. Since $\lim_{h\to0}[f(x+h)-f(x-h)] = f(x)-f(x)=0$ (f is differentiable and hence continuous) and $\lim_{h\to0}2h=0$, we use l'Hospital's Rule:

$$\lim_{h\to0}\frac{f(x+h)-f(x-h)}{2h} \overset{\text{H}}{=} \lim_{h\to0}\frac{f'(x+h)(1)-f'(x-h)(-1)}{2} = \frac{f'(x)+f'(x)}{2} = \frac{2f'(x)}{2} = f'(x)$$

$\dfrac{f(x+h)-f(x-h)}{2h}$ is the slope of the secant line between $(x-h,f(x-h))$ and $(x+h,f(x+h))$. As $h\to0$, this line gets closer to the tangent line and its slope approaches $f'(x)$.



82. Since $\lim_{h\to 0}\,[f(x+h) - 2f(x) + f(x-h)] = f(x) - 2f(x) + f(x) = 0$ [f is differentiable and hence continuous] and $\lim_{h\to 0} h^2 = 0$, we can apply l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \overset{\text{H}}{=} \lim_{h\to 0}\frac{f'(x+h) - f'(x-h)}{2h} = f''(x)$$

At the last step, we have applied the result of Exercise 81 to $f'(x)$.

83. (a) We show that $\lim_{x\to 0}\dfrac{f(x)}{x^n} = 0$ for every integer $n \geq 0$. Let $y = \dfrac{1}{x^2}$. Then

$$\lim_{x\to 0}\frac{f(x)}{x^{2n}} = \lim_{x\to 0}\frac{e^{-1/x^2}}{(x^2)^n} = \lim_{y\to\infty}\frac{y^n}{e^y} \overset{\text{H}}{=} \lim_{y\to\infty}\frac{ny^{n-1}}{e^y} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim_{y\to\infty}\frac{n!}{e^y} = 0 \;\Rightarrow$$

$$\lim_{x\to 0}\frac{f(x)}{x^n} = \lim_{x\to 0} x^n\frac{f(x)}{x^{2n}} = \lim_{x\to 0} x^n \lim_{x\to 0}\frac{f(x)}{x^{2n}} = 0. \text{ Thus, } f'(0) = \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{f(x)}{x} = 0.$$

(b) Using the Chain Rule and the Quotient Rule we see that $f^{(n)}(x)$ exists for $x \neq 0$. In fact, we prove by induction that for each $n \geq 0$, there is a polynomial p_n and a non-negative integer k_n with $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$. This is true for $n = 0$; suppose it is true for the nth derivative. Then $f'(x) = f(x)(2/x^3)$, so

$$\begin{aligned} f^{(n+1)}(x) &= \left[x^{k_n}\left[p_n'(x)\,f(x) + p_n(x)f'(x)\right] - k_n x^{k_n-1}p_n(x)\,f(x)\right]x^{-2k_n} \\ &= \left[x^{k_n}p_n'(x) + p_n(x)\left(2/x^3\right) - k_n x^{k_n-1}p_n(x)\right]f(x)x^{-2k_n} \\ &= \left[x^{k_n+3}p_n'(x) + 2p_n(x) - k_n x^{k_n+2}\,p_n(x)\right]f(x)x^{-(2k_n+3)} \end{aligned}$$

which has the desired form.

Now we show by induction that $f^{(n)}(0) = 0$ for all n. By part (a), $f'(0) = 0$. Suppose that $f^{(n)}(0) = 0$. Then

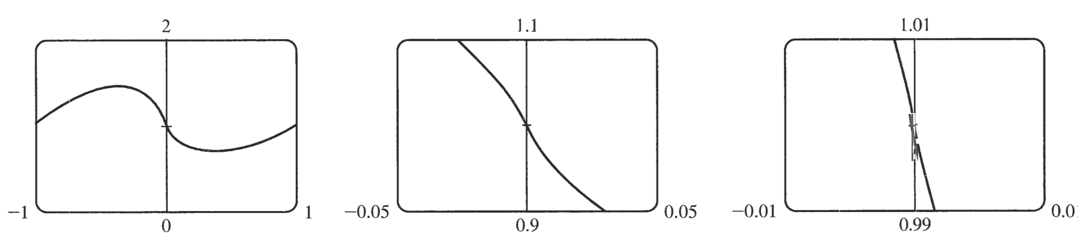
$$\begin{aligned} f^{(n+1)}(0) &= \lim_{x\to 0}\frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x\to 0}\frac{f^{(n)}(x)}{x} = \lim_{x\to 0}\frac{p_n(x)\,f(x)/x^{k_n}}{x} = \lim_{x\to 0}\frac{p_n(x)\,f(x)}{x^{k_n+1}} \\ &= \lim_{x\to 0} p_n(x)\lim_{x\to 0}\frac{f(x)}{x^{k_n+1}} = p_n(0)\cdot 0 = 0 \end{aligned}$$

84. (a) For f to be continuous, we need $\lim_{x\to 0} f(x) = f(0) = 1$. We note that for $x \neq 0$, $\ln f(x) = \ln|x|^x = x\ln|x|$.

So $\lim_{x\to 0}\ln f(x) = \lim_{x\to 0} x\ln|x| = \lim_{x\to 0}\dfrac{\ln|x|}{1/x} \overset{\text{H}}{=} \lim_{x\to 0}\dfrac{1/x}{-1/x^2} = 0$. Therefore, $\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\ln f(x)} = e^0 = 1$.

So f is continuous at 0.

(b) From the graphs, it appears that f is differentiable at 0.



(c) To find f', we use logarithmic differentiation: $\ln f(x) = x \ln|x| \Rightarrow \dfrac{f'(x)}{f(x)} = x\left(\dfrac{1}{x}\right) + \ln|x| \Rightarrow$

$f'(x) = f(x)(1 + \ln|x|) = |x|^x(1 + \ln|x|)$, $x \neq 0$. Now $f'(x) \to -\infty$ as $x \to 0$ [since $|x|^x \to 1$ and $(1 + \ln|x|) \to -\infty$], so the curve has a vertical tangent at $(0, 1)$ and is therefore not differentiable there. The fact cannot be seen in the graphs in part (b) because $\ln|x| \to -\infty$ very slowly as $x \to 0$.