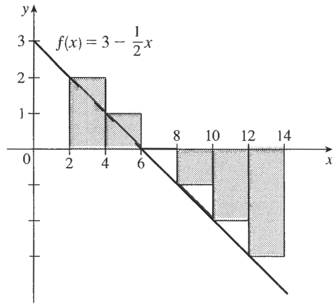
1. $f(x) = 3 - \frac{1}{2}x,\ 2 \le x \le 14.\ \Delta x = \dfrac{b-a}{n} = \dfrac{14-2}{6} = 2.$




Since we are using left endpoints, $x_i^* = x_{i-1}$.

$$L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x$$

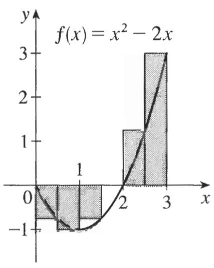
$$= (\Delta x)\,[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$$

$$= 2[f(2) + f(4) + f(6) + f(8) + f(10) + f(12)]$$

$$= 2[2 + 1 + 0 + (-1) + (-2) + (-3)] = 2(-3) = -6$$

The Riemann sum represents the sum of the areas of the two rectangles above the $x$-axis minus the sum of the areas of the three rectangles below the $x$-axis; that is, the *net area* of the rectangles with respect to the $x$-axis.

**2.** $f(x) = x^2 - 2x$, $0 \le x \le 3$. $\Delta x = \dfrac{b-a}{n} = \dfrac{3-0}{6} = \dfrac{1}{2}$.

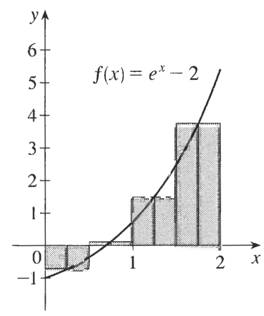



Since we are using right endpoints, $x_i^* = x_i$.

$$\begin{aligned} R_6 &= \sum_{i=1}^{6} f(x_i)\,\Delta x \\ &= (\Delta x)\,[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)] \\ &= \tfrac{1}{2}\left[f\left(\tfrac{1}{2}\right) + f(1) + f\left(\tfrac{3}{2}\right) + f(2) + f\left(\tfrac{5}{2}\right) + f(3)\right] \\ &= \tfrac{1}{2}\left(-\tfrac{3}{4} - 1 - \tfrac{3}{4} + 0 + \tfrac{5}{4} + 3\right) = \tfrac{1}{2}\left(\tfrac{7}{4}\right) = \tfrac{7}{8} \end{aligned}$$

The Riemann sum represents the sum of the areas of the two rectangles above the $x$-axis minus the sum of the areas of the three rectangles below the $x$-axis; that is, the *net area* of the rectangles with respect to the $x$-axis.

**3.** $f(x) = e^x - 2$, $0 \le x \le 2$. $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{4} = \dfrac{1}{2}$.

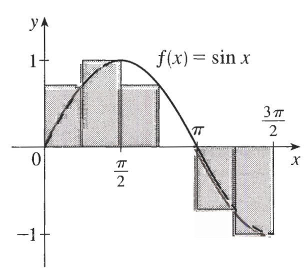



Since we are using midpoints, $x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$.

$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = (\Delta x)\,[f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4)] \\ &= \tfrac{1}{2}\left[f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right) + f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right)\right] \\ &= \tfrac{1}{2}\left[(e^{1/4} - 2) + (e^{3/4} - 2) + (e^{5/4} - 2) + (e^{7/4} - 2)\right] \\ &\approx 2.322986 \end{aligned}$$

The Riemann sum represents the sum of the areas of the three rectangles above the $x$-axis minus the area of the rectangle below the $x$-axis; that is, the *net area* of the rectangles with respect to the $x$-axis.

**4.** (a) $f(x) = \sin x$, $0 \le x \le \dfrac{3\pi}{2}$. $\Delta x = \dfrac{b-a}{n} = \dfrac{\frac{3\pi}{2} - 0}{6} = \dfrac{\pi}{4}$.

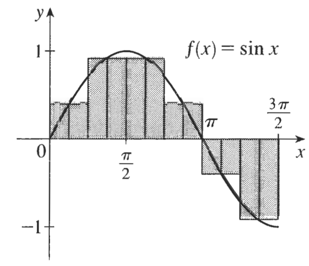



Since we are using right endpoints, $x_i^* = x_i$.

$$\begin{aligned} R_6 &= \sum_{i=1}^{6} f(x_i)\,\Delta x \\ &= (\Delta x)[\,f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)] \\ &= \frac{\pi}{4}\left[f\left(\frac{\pi}{4}\right) + f\left(\frac{2\pi}{4}\right) + f\left(\frac{3\pi}{4}\right) + f\left(\frac{4\pi}{4}\right) + f\left(\frac{5\pi}{4}\right) + f\left(\frac{6\pi}{4}\right)\right] \\ &= \frac{\pi}{4}\left(\sin\frac{\pi}{4} + \sin\frac{\pi}{2} + \sin\frac{3\pi}{4} + \sin\pi + \sin\frac{5\pi}{4} + \sin\frac{3\pi}{2}\right) \\ &= \frac{\pi}{4}\left(\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{2}}{2} - 1\right) = \frac{\pi\sqrt{2}}{8} \approx 0.555360 \end{aligned}$$

The Riemann sum represents the sum of the areas of the three rectangles above the $x$-axis minus the sum of the areas of the two rectangles below the $x$-axis.

(b) Since we are using midpoints, $x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$.




$$\begin{aligned} M_6 &= \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x \\ &= (\Delta x)\left[ f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4) + f(\overline{x}_5) + f(\overline{x}_6)\right] \\ &= \frac{\pi}{4}\left[ f\left(\frac{\pi}{8}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{5\pi}{8}\right) + f\left(\frac{7\pi}{8}\right) + f\left(\frac{9\pi}{8}\right) + f\left(\frac{11\pi}{8}\right)\right] \\ &= \frac{\pi}{4}\left( \sin\frac{\pi}{8} + \sin\frac{3\pi}{8} + \sin\frac{5\pi}{8} + \sin\frac{7\pi}{8} + \sin\frac{9\pi}{8} + \sin\frac{11\pi}{8}\right) \\ &\approx \tfrac{\pi}{4}(1.306563) \approx 1.026172 \end{aligned}$$

The Riemann sum represents the sum of the areas of the four rectangles above the $x$-axis minus the sum of the areas of the two rectangles below the $x$-axis. Note that the Riemann sum has the same value as the sum of the areas of the first two rectangles.

**5.** $\Delta x = (b-a)/n = (8-0)/4 = 8/4 = 2$.

(a) Using the right endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(x_i)\,\Delta x = 2[f(2) + f(4) + f(6) + f(8)] \approx 2[1 + 2 + (-2) + 1] = 4.$$

(b) Using the left endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(x_{i-1})\,\Delta x = 2[f(0) + f(2) + f(4) + f(6)] \approx 2[2 + 1 + 2 + (-2)] = 6.$$

(c) Using the midpoint of each subinterval to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = 2[f(1) + f(3) + f(5) + f(7)] \approx 2[3 + 2 + 1 + (-1)] = 10.$$

**6.** (a) Using the right endpoints to approximate $\int_{-3}^3 g(x)\,dx$, we have

$$\sum_{i=1}^{6} g(x_i)\,\Delta x = 1[g(-2) + g(-1) + g(0) + g(1) + g(2) + g(3)] \approx 1 - 0.5 - 1.5 - 1.5 - 0.5 + 2.5 = -0.5.$$

(b) Using the left endpoints to approximate $\int_{-3}^3 g(x)\,dx$, we have

$$\sum_{i=1}^{6} g(x_{i-1})\,\Delta x = 1[g(-3) + g(-2) + g(-1) + g(0) + g(1) + g(2)] \approx 2 + 1 - 0.5 - 1.5 - 1.5 - 0.5 = -1.$$

(c) Using the midpoint of each subinterval to approximate $\int_{-3}^3 g(x)\,dx$, we have

$$\begin{aligned} \sum_{i=1}^{6} g(\overline{x}_i)\,\Delta x &= 1[g(-2.5) + g(-1.5) + g(-0.5) + g(0.5) + g(1.5) + g(2.5)] \\ &\approx 1.5 + 0 - 1 - 1.75 - 1 + 0.5 = -1.75 \end{aligned}$$

**7.** Since $f$ is increasing, $L_5 \le \int_0^{25} f(x)\,dx \le R_5$.

$$\begin{aligned} \text{Lower estimate} = L_5 &= \sum_{i=1}^{5} f(x_{i-1})\,\Delta x = 5[f(0) + f(5) + f(10) + f(15) + f(20)] \\ &= 5(-42 - 37 - 25 - 6 + 15) = 5(-95) = -475 \end{aligned}$$

$$\begin{aligned} \text{Upper estimate} = R_5 &= \sum_{i=1}^{5} f(x_i)\,\Delta x = 5[f(5) + f(10) + f(15) + f(20) + f(25)] \\ &= 5(-37 - 25 - 6 + 15 + 36) = 5(-17) = -85 \end{aligned}$$

**8.** (a) Using the right endpoints to approximate $\int_3^9 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(x_i)\,\Delta x = 2[f(5) + f(7) + f(9)] = 2\,(-0.6 + 0.9 + 1.8) = 4.2.$$

Since $f$ is *increasing*, using *right* endpoints gives an *overestimate*.

(b) Using the left endpoints to approximate $\int_3^9 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(x_{i-1})\,\Delta x = 2[f(3) + f(5) + f(7)] = 2\,(-3.4 - 0.6 + 0.9) = -6.2.$$

Since $f$ is *increasing*, using *left* endpoints gives an *underestimate*.

(c) Using the midpoint of each interval to approximate $\int_3^9 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(\overline{x}_i)\,\Delta x = 2[f(4) + f(6) + f(8)] = 2\,(-2.1 + 0.3 + 1.4) = -0.8.$$

We cannot say anything about the midpoint estimate compared to the exact value of the integral.

**9.** $\Delta x = (10 - 2)/4 = 2$, so the endpoints are 2, 4, 6, 8, and 10, and the midpoints are 3, 5, 7, and 9. The Midpoint Rule gives $\int_2^{10} \sqrt{x^3 + 1}\,dx \approx \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = 2\left(\sqrt{3^3+1} + \sqrt{5^3+1} + \sqrt{7^3+1} + \sqrt{9^3+1}\right) \approx 124.1644.$

**10.** $\Delta x = (\pi/2 - 0)/4 = \frac{\pi}{8}$, so the endpoints are 0, $\frac{\pi}{8}$, $\frac{\pi}{4}$, $\frac{3\pi}{8}$, and $\frac{\pi}{2}$, and the midpoints are $\frac{\pi}{16}$, $\frac{3\pi}{16}$, $\frac{5\pi}{16}$, and $\frac{7\pi}{16}$. The Midpoint Rule gives

$$\int_0^{\pi/2} \cos^4 x\,dx \approx \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = \frac{\pi}{8}\left[\cos^4\left(\frac{\pi}{16}\right) + \cos^4\left(\frac{3\pi}{16}\right) + \cos^4\left(\frac{5\pi}{16}\right) + \cos^4\left(\frac{7\pi}{16}\right)\right] = \frac{\pi}{8}\left(\frac{3}{2}\right) \approx 0.5890.$$

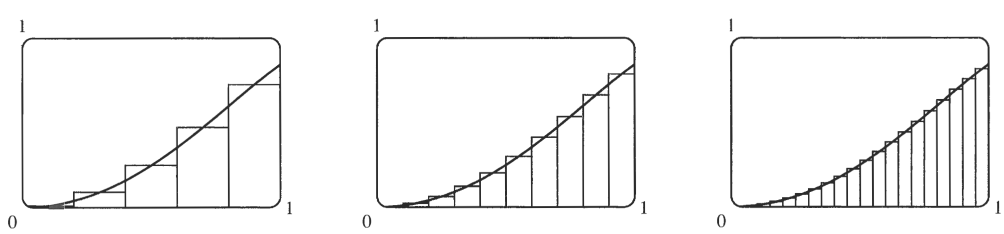
**11.** $\Delta x = (1 - 0)/5 = 0.2$, so the endpoints are 0, 0.2, 0.4, 0.6, 0.8, and 1, and the midpoints are 0.1, 0.3, 0.5, 0.7, and 0.9. The Midpoint Rule gives

$$\int_0^1 \sin(x^2)\,dx \approx \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x = 0.2\left[\sin(0.1)^2 + \sin(0.3)^2 + \sin(0.5)^2 + \sin(0.7)^2 + \sin(0.9)^2\right] \approx 0.3084.$$

**12.** $\Delta x = (5 - 1)/4 = 1$, so the endpoints are 1, 2, 3, 4, and 5, and the midpoints are 1.5, 2.5, 3.5, and 4.5. The Midpoint Rule gives

$$\int_1^5 x^2 e^{-x}\,dx \approx \sum_{n=1}^{4} f(\overline{x}_i)\,\Delta x = 1\left[(1.5)^2 e^{-1.5} + (2.5)^2 e^{-2.5} + (3.5)^2 e^{-3.5} + (4.5)^2 e^{-4.5}\right] \approx 1.6099.$$

**13.** In Maple, we use the command `with(student);` to load the sum and box commands, then `m:=middlesum(sin(x^2),x=0..1,5);` which gives us the sum in summation notation, then `M:=evalf(m);` which gives $M_5 \approx 0.30843908$, confirming the result of Exercise 11. The command `middlebox(sin(x^2),x=0..1,5)` generates the graph. Repeating for $n = 10$ and $n = 20$ gives $M_{10} \approx 0.30981629$ and $M_{20} \approx 0.31015563$.



**14.** See the solution to Exercise 5.1.7 for a possible algorithm to calculate the sums. With $\Delta x = (1-0)/100 = 0.01$ and subinterval endpoints 1, 1.01, 1.02, ..., 1.99, 2, we calculate that the left Riemann sum is

$L_{100} = \sum\limits_{i=1}^{100} \sin(x_{i-1}^2)\,\Delta x \approx 0.30607$, and the right Riemann sum is $R_{100} = \sum\limits_{i=1}^{100} \sin(x_i^2)\,\Delta x \approx 0.31448$.

Since $f(x) = \sin(x^2)$ is an increasing function, we must have $L_{100} \le \int_0^1 \sin(x^2)\,dx \le R_{100}$, so $0.306 < L_{100} \le \int_0^1 \sin(x^2)\,dx \le R_{100} < 0.315$. Therefore, the approximate value $0.3084 \approx 0.31$ in Exercise 11 must be accurate to two decimal places.

**15.** We'll create the table of values to approximate $\int_0^\pi \sin x\,dx$ by using the program in the solution to Exercise 5.1.7 with $Y_1 = \sin x$, Xmin $= 0$, Xmax $= \pi$, and $n = 5, 10, 50$, and 100.

| $n$ | $R_n$ |
|---|---|
| 5 | 1.933766 |
| 10 | 1.983524 |
| 50 | 1.999342 |
| 100 | 1.999836 |

The values of $R_n$ appear to be approaching 2.

**16.** $\int_0^2 e^{-x^2}\,dx$ with $n = 5, 10, 50$, and 100.

| $n$ | $L_n$ | $R_n$ |
|---|---|---|
| 5 | 1.077467 | 0.684794 |
| 10 | 0.980007 | 0.783670 |
| 50 | 0.901705 | 0.862438 |
| 100 | 0.891896 | 0.872262 |

The value of the integral lies between 0.872 and 0.892. Note that $f(x) = e^{-x^2}$ is decreasing on $(0, 2)$. We cannot make a similar statement for $\int_{-1}^2 e^{-x^2}\,dx$ since $f$ is increasing on $(-1, 0)$.

**17.** On $[2, 6]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} x_i \ln(1 + x_i^2)\,\Delta x = \int_2^6 x\ln(1+x^2)\,dx$.

**18.** On $[\pi, 2\pi]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \dfrac{\cos x_i}{x_i}\,\Delta x = \displaystyle\int_\pi^{2\pi} \frac{\cos x}{x}\,dx$.

**19.** On $[1, 8]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \sqrt{2x_i^* + (x_i^*)^2}\,\Delta x = \int_1^8 \sqrt{2x + x^2}\,dx$.

**20.** On $[0, 2]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} [4 - 3(x_i^*)^2 + 6(x_i^*)^5]\,\Delta x = \int_0^2 (4 - 3x^2 + 6x^5)\,dx$.

**21.** Note that $\Delta x = \dfrac{5-(-1)}{n} = \dfrac{6}{n}$ and $x_i = -1 + i\,\Delta x = -1 + \dfrac{6i}{n}$.

$$\begin{aligned}
\int_{-1}^{5}(1+3x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[1+3\left(-1+\frac{6i}{n}\right)\right]\frac{6}{n} = \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n}\left[-2+\frac{18i}{n}\right] \\
&= \lim_{n\to\infty}\frac{6}{n}\left[\sum_{i=1}^{n}(-2)+\sum_{i=1}^{n}\frac{18i}{n}\right] = \lim_{n\to\infty}\frac{6}{n}\left[-2n+\frac{18}{n}\sum_{i=1}^{n} i\right] \\
&= \lim_{n\to\infty}\frac{6}{n}\left[-2n+\frac{18}{n}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[-12+\frac{108}{n^2}\cdot\frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty}\left[-12+54\frac{n+1}{n}\right] = \lim_{n\to\infty}\left[-12+54\left(1+\frac{1}{n}\right)\right] = -12+54\cdot 1 = 42
\end{aligned}$$

**22.** $\displaystyle\int_1^4 (x^2+2x-5)\,dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x \qquad [\Delta x = 3/n \text{ and } x_i = 1+3i/n]$

$$= \lim_{n\to\infty}\sum_{i=1}^n\left[\left(1+\frac{3i}{n}\right)^2 + 2\left(1+\frac{3i}{n}\right) - 5\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\sum_{i=1}^n\left(1+\frac{6i}{n}+\frac{9i^2}{n^2}+2+\frac{6i}{n}-5\right)\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\sum_{i=1}^n\left(\frac{9}{n^2}\cdot i^2+\frac{12}{n}\cdot i-2\right)\right] = \lim_{n\to\infty}\frac{3}{n}\left[\frac{9}{n^2}\sum_{i=1}^n i^2+\frac{12}{n}\sum_{i=1}^n i-\sum_{i=1}^n 2\right]$$

$$= \lim_{n\to\infty}\left(\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{36}{n^2}\cdot\frac{n(n+1)}{2}-\frac{6}{n}\cdot n\right)$$

$$= \lim_{n\to\infty}\left(\frac{9}{2}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}+18\cdot\frac{n+1}{n}-6\right)$$

$$= \lim_{n\to\infty}\left[\frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+18\left(1+\frac{1}{n}\right)-6\right] = \tfrac{9}{2}\cdot 1\cdot 2+18\cdot 1-6 = 21$$

**23.** Note that $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$$\int_0^2 (2-x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^n\left(2-\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n}\left[\sum_{i=1}^n 2-\frac{4}{n^2}\sum_{i=1}^n i^2\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\left(2n-\frac{4}{n^2}\sum_{i=1}^n i^2\right) = \lim_{n\to\infty}\left[4-\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left(4-\frac{4}{3}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right) = \lim_{n\to\infty}\left[4-\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 4-\tfrac{4}{3}\cdot 1\cdot 2 = \tfrac{4}{3}$$

**24.** $\displaystyle\int_0^5 (1+2x^3)\,dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\,\Delta x \qquad [\Delta x = 5/n \text{ and } x_i = 5i/n]$

$$= \lim_{n\to\infty}\sum_{i=1}^n\left(1+2\cdot\frac{125i^3}{n^3}\right)\left(\frac{5}{n}\right) = \lim_{n\to\infty}\frac{5}{n}\left[\sum_{i=1}^n 1+\frac{250}{n^3}\sum_{i=1}^n i^3\right]$$

$$= \lim_{n\to\infty}\frac{5}{n}\left(1\cdot n+\frac{250}{n^3}\sum_{i=1}^n i^3\right) = \lim_{n\to\infty}\left[5+\frac{1250}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty}\left[5+312.5\cdot\frac{(n+1)^2}{n^2}\right] = \lim_{n\to\infty}\left[5+312.5\left(1+\frac{1}{n}\right)^2\right]$$

$$= 5+312.5 = 317.5$$

**25.** Note that $\Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}$ and $x_i = 1 + i\,\Delta x = 1 + i(1/n) = 1 + i/n$.

$$\int_1^2 x^3\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^n \left(1+\frac{i}{n}\right)^3 \left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{n+i}{n}\right)^3$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^n (n^3 + 3n^2 i + 3ni^2 + i^3) = \lim_{n\to\infty} \frac{1}{n^4} \left[\sum_{i=1}^n n^3 + \sum_{i=1}^n 3n^2 i + \sum_{i=1}^n 3ni^2 + \sum_{i=1}^n i^3\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left[n\cdot n^3 + 3n^2 \sum_{i=1}^n i + 3n \sum_{i=1}^n i^2 + \sum_{i=1}^n i^3\right]$$

$$= \lim_{n\to\infty} \left[1 + \frac{3}{n^2}\cdot\frac{n(n+1)}{2} + \frac{3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty} \left[1 + \frac{3}{2}\cdot\frac{n+1}{n} + \frac{1}{2}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} + \frac{1}{4}\cdot\frac{(n+1)^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \left[1 + \frac{3}{2}\left(1+\frac{1}{n}\right) + \frac{1}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{1}{4}\left(1+\frac{1}{n}\right)^2\right] = 1 + \frac{3}{2} + \tfrac{1}{2}\cdot 2 + \tfrac{1}{4} = 3.75$$

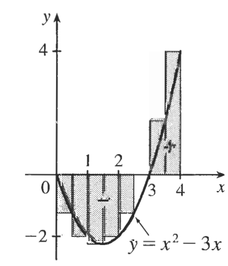
**26.** (a) $\Delta x = (4-0)/8 = 0.5$ and $x_i^* = x_i = 0.5i$.

$$\int_0^4 (x^2 - 3x)\,dx \approx \sum_{i=1}^8 f(x_i^*)\,\Delta x$$

$$= 0.5\{[0.5^2 - 3(0.5)] + [1.0^2 - 3(1.0)] + \cdots + [3.5^2 - 3(3.5)] + [4.0^2 - 3(4.0)]\}$$

$$= \tfrac{1}{2}\left(-\tfrac{5}{4} - 2 - \tfrac{9}{4} - 2 - \tfrac{5}{4} + 0 + \tfrac{7}{4} + 4\right) = -1.5$$

(b)




(c)
$$\int_0^4 (x^2 - 3x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n \left[\left(\frac{4i}{n}\right)^2 - 3\left(\frac{4i}{n}\right)\right]\left(\frac{4}{n}\right)$$

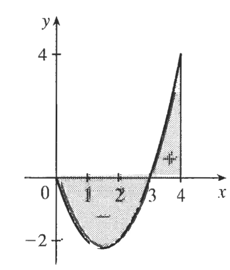
$$= \lim_{n\to\infty} \frac{4}{n}\left[\frac{16}{n^2}\sum_{i=1}^n i^2 - \frac{12}{n}\sum_{i=1}^n i\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{48}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 24\left(1+\frac{1}{n}\right)\right]$$

$$= \tfrac{32}{3}\cdot 2 - 24 = -\tfrac{8}{3}$$

(d) $\int_0^4 (x^2 - 3x)\,dx = A_1 - A_2$, where $A_1$ is the area marked $+$ and $A_2$ is the area marked $-$.




**27.**
$$\int_a^b x\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^n \left[a + \frac{b-a}{n} i\right] = \lim_{n\to\infty} \left[\frac{a(b-a)}{n} \sum_{i=1}^n 1 + \frac{(b-a)^2}{n^2} \sum_{i=1}^n i\right]$$

$$= \lim_{n\to\infty} \left[\frac{a(b-a)}{n} n + \frac{(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2}\right] = a(b-a) + \lim_{n\to\infty} \frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)$$

$$= a(b-a) + \tfrac{1}{2}(b-a)^2 = (b-a)\left(a + \tfrac{1}{2}b - \tfrac{1}{2}a\right) = (b-a)\tfrac{1}{2}(b+a) = \tfrac{1}{2}(b^2 - a^2)$$

**28.** $$\int_a^b x^2\,dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^n\left[a+\frac{b-a}{n}i\right]^2 = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^n\left[a^2+2a\frac{b-a}{n}i+\frac{(b-a)^2}{n^2}i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\sum_{i=1}^n i^2+\frac{2a\,(b-a)^2}{n^2}\sum_{i=1}^n i+\frac{a^2\,(b-a)}{n}\sum_{i=1}^n 1\right]$$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\frac{n\,(n+1)\,(2n+1)}{6}+\frac{2a\,(b-a)^2}{n^2}\frac{n\,(n+1)}{2}+\frac{a^2\,(b-a)}{n}n\right]$$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{6}\cdot 1\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+a\,(b-a)^2\cdot 1\cdot\left(1+\frac{1}{n}\right)+a^2\,(b-a)\right]$$

$$= \frac{(b-a)^3}{3}+a\,(b-a)^2+a^2\,(b-a) = \frac{b^3-3ab^2+3a^2b-a^3}{3}+ab^2-2a^2b+a^3+a^2b-a^3$$

$$= \frac{b^3}{3}-\frac{a^3}{3}-ab^2+a^2b+ab^2-a^2b = \frac{b^3-a^3}{3}$$

**29.** $f(x) = \dfrac{x}{1+x^5}$, $a = 2$, $b = 6$, and $\Delta x = \dfrac{6-2}{n} = \dfrac{4}{n}$. Using Equation 4, we get $x_i^* = x_i = 2 + i\,\Delta x = 2 + \dfrac{4i}{n}$,

so $$\int_2^6 \frac{x}{1+x^5}\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^n \frac{2+\dfrac{4i}{n}}{1+\left(2+\dfrac{4i}{n}\right)^5}\cdot\frac{4}{n}.$$

**30.** $\Delta x = \dfrac{10-1}{n} = \dfrac{9}{n}$ and $x_i = 1 + i\,\Delta x = 1 + \dfrac{9i}{n}$, so

$$\int_1^{10}(x-4\ln x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^n\left[\left(1+\frac{9i}{n}\right)-4\ln\left(1+\frac{9i}{n}\right)\right]\cdot\frac{9}{n}.$$

**31.** $\Delta x = (\pi - 0)/n = \pi/n$ and $x_i^* = x_i = \pi i/n$.

$$\int_0^\pi \sin 5x\,dx = \lim_{n\to\infty}\sum_{i=1}^n(\sin 5x_i)\left(\frac{\pi}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^n\left(\sin\frac{5\pi i}{n}\right)\frac{\pi}{n} \overset{\text{CAS}}{=} \pi\lim_{n\to\infty}\frac{1}{n}\cot\left(\frac{5\pi}{2n}\right) \overset{\text{CAS}}{=} \pi\left(\frac{2}{5\pi}\right) = \frac{2}{5}$$

**32.** $\Delta x = (10-2)/n = 8/n$ and $x_i^* = x_i = 2 + 8i/n$.

$$\int_2^{10} x^6\,dx = \lim_{n\to\infty}\sum_{i=1}^n\left(2+\frac{8i}{n}\right)^6\left(\frac{8}{n}\right) = 8\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\left(2+\frac{8i}{n}\right)^6$$

$$\overset{\text{CAS}}{=} 8\lim_{n\to\infty}\frac{1}{n}\cdot\frac{64(58{,}593n^6+164{,}052n^5+131{,}208n^4-27{,}776n^2+2048)}{21n^5}$$

$$\overset{\text{CAS}}{=} 8\left(\frac{1{,}249{,}984}{7}\right) = \frac{9{,}999{,}872}{7} \approx 1{,}428{,}553.1$$

**33.** (a) Think of $\int_0^2 f(x)\,dx$ as the area of a trapezoid with bases 1 and 3 and height 2. The area of a trapezoid is $A = \frac{1}{2}(b+B)h$, so $\int_0^2 f(x)\,dx = \frac{1}{2}(1+3)2 = 4$.

(b) $\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^5 f(x)\,dx$

trapezoid, rectangle, triangle

$= \frac{1}{2}(1+3)2 + 3\cdot 1 + \frac{1}{2}\cdot 2\cdot 3 = 4+3+3 = 10$

(c) $\int_5^7 f(x)\,dx$ is the negative of the area of the triangle with base 2 and height 3. $\int_5^7 f(x)\,dx = -\frac{1}{2}\cdot 2\cdot 3 = -3$.

(d) $\int_7^9 f(x)\,dx$ is the negative of the area of a trapezoid with bases 3 and 2 and height 2, so it equals

$-\frac{1}{2}(B+b)h = -\frac{1}{2}(3+2)2 = -5$. Thus,

$\int_0^9 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx + \int_7^9 f(x)\,dx = 10 + (-3) + (-5) = 2.$

**34.** (a) $\int_0^2 g(x)\,dx = \frac{1}{2} \cdot 4 \cdot 2 = 4$ [area of a triangle]

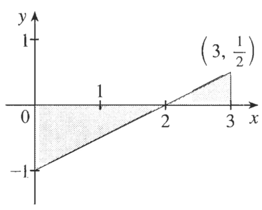
(b) $\int_2^6 g(x)\,dx = -\frac{1}{2}\pi(2)^2 = -2\pi$ [negative of the area of a semicircle]

(c) $\int_6^7 g(x)\,dx = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ [area of a triangle]

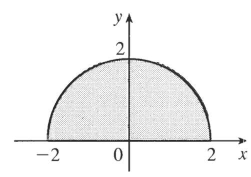
$\int_0^7 g(x)\,dx = \int_0^2 g(x)\,dx + \int_2^6 g(x)\,dx + \int_6^7 g(x)\,dx = 4 - 2\pi + \frac{1}{2} = 4.5 - 2\pi$

**35.** $\int_0^3 \left(\frac{1}{2}x - 1\right) dx$ can be interpreted as the area of the triangle above the $x$-axis minus the area of the triangle below the $x$-axis; that is,

$\frac{1}{2}(1)\left(\frac{1}{2}\right) - \frac{1}{2}(2)(1) = \frac{1}{4} - 1 = -\frac{3}{4}$.

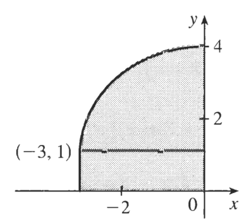



**36.** $\int_{-2}^2 \sqrt{4-x^2}\,dx$ can be interpreted as the area under the graph of $f(x) = \sqrt{4-x^2}$ between $x=-2$ and $x=2$. This is equal to half the area of the circle with radius 2, so $\int_{-2}^2 \sqrt{4-x^2}\,dx = \frac{1}{2}\pi \cdot 2^2 = 2\pi$.



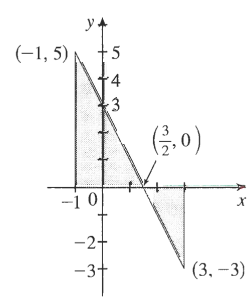

**37.** $\int_{-3}^0 \left(1+\sqrt{9-x^2}\right) dx$ can be interpreted as the area under the graph of $f(x) = 1+\sqrt{9-x^2}$ between $x=-3$ and $x=0$. This is equal to one-quarter the area of the circle with radius 3, plus the area of the rectangle, so

$\int_{-3}^0 \left(1+\sqrt{9-x^2}\right) dx = \frac{1}{4}\pi \cdot 3^2 + 1 \cdot 3 = 3 + \frac{9}{4}\pi$.

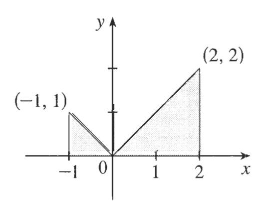



**38.** $\int_{-1}^3 (3-2x)\,dx$ can be interpreted as the area of the triangle above the $x$-axis minus the area of the triangle below the $x$-axis; that is,

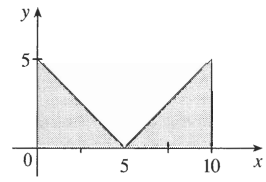
$\frac{1}{2}\left(\frac{5}{2}\right)(5) - \frac{1}{2}\left(\frac{3}{2}\right)(3) = \frac{25}{4} - \frac{9}{4} = 4$.




**39.** $\int_{-1}^2 |x|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = \frac{1}{2} + \frac{4}{2} = \frac{5}{2}$.



**40.** $\int_0^{10} |x-5|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $2\left(\frac{1}{2}\right)(5)(5) = 25$.




**41.** $\int_\pi^\pi \sin^2 x \cos^4 x\,dx = 0$ since the limits of intergration are equal.

**42.** $\int_1^0 3u\sqrt{u^2+4}\,du = -\int_0^1 3u\sqrt{u^2+4}\,du$ [because we reversed the limits of integration]

$= -\int_0^1 3x\sqrt{x^2+4}\,dx$ [we can use any letter without changing the value of the inegral]

$= -(5\sqrt{5}-8)$ [given value]

$= 8 - 5\sqrt{5}$

**43.** $\int_0^1 (5-6x^2)\,dx = \int_0^1 5\,dx - 6\int_0^1 x^2\,dx = 5(1-0) - 6\left(\frac{1}{3}\right) = 5-2 = 3$

**44.** $\int_1^3 (2e^x - 1)\,dx = 2\int_1^3 e^x\,dx - \int_1^3 1\,dx = 2(e^3-e) - 1(3-1) = 2e^3 - 2e - 2$

**45.** $\int_1^3 e^{x+2}\,dx = \int_1^3 e^x \cdot e^2\,dx = e^2\int_1^3 e^x\,dx = e^2(e^3-e) = e^5 - e^3$

**46.** $\int_0^{\pi/2} (2\cos x - 5x)\,dx = \int_0^{\pi/2} 2\cos x\,dx - \int_0^{\pi/2} 5x\,dx = 2\int_0^{\pi/2}\cos x\,dx - 5\int_0^{\pi/2} x\,dx$

$$= 2(1) - 5\frac{(\pi/2)^2 - 0^2}{2} = 2 - \frac{5\pi^2}{8}$$

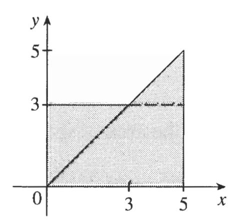
**47.** $\int_{-2}^2 f(x)\,dx + \int_2^5 f(x)\,dx - \int_{-2}^{-1} f(x)\,dx = \int_{-2}^5 f(x)\,dx + \int_{-1}^{-2} f(x)\,dx$ [by Property 5 and reversing limits]

$= \int_{-1}^5 f(x)\,dx$ [Property 5]

**48.** $\int_1^4 f(x)\,dx = \int_1^5 f(x)\,dx - \int_4^5 f(x)\,dx = 12 - 3.6 = 8.4$

**49.** $\int_0^9 [2f(x) + 3g(x)]\,dx = 2\int_0^9 f(x)\,dx + 3\int_0^9 g(x)\,dx = 2(37) + 3(16) = 122$

**50.** If $f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3 \end{cases}$, then $\int_0^5 f(x)\,dx$ can be interpreted as the area of the shaded region, which consists of a 5-by-3 rectangle surmounted by an isosceles right triangle whose legs have length 2. Thus, $\int_0^5 f(x)\,dx = 5(3) + \frac{1}{2}(2)(2) = 17$.




**51.** Using Integral Comparison Property 8, $m \leq f(x) \leq M \;\Rightarrow\; m(2-0) \leq \int_0^2 f(x)\,dx \leq M(2-0) \;\Rightarrow$ $2m \leq \int_0^2 f(x)\,dx \leq 2M$.

**52.** $x^2 \leq x$ on $[0,1]$, so $\sqrt{1+x^2} \leq \sqrt{1+x}$ on $[0,1]$. Hence, $\int_0^1 \sqrt{1+x^2}\,dx \leq \int_0^1 \sqrt{1+x}\,dx$ [Property 7].

**53.** If $-1 \leq x \leq 1$, then $0 \leq x^2 \leq 1$ and $1 \leq 1+x^2 \leq 2$, so $1 \leq \sqrt{1+x^2} \leq \sqrt{2}$ and $1[1-(-1)] \leq \int_{-1}^1 \sqrt{1+x^2}\,dx \leq \sqrt{2}\,[1-(-1)]$ [Property 8]; that is, $2 \leq \int_{-1}^1 \sqrt{1+x^2}\,dx \leq 2\sqrt{2}$.

**54.** If $\frac{\pi}{6} \leq x \leq \frac{\pi}{4}$, then $\cos\frac{\pi}{6} \geq \cos x \geq \cos\frac{\pi}{4}$ and $\frac{\sqrt{2}}{2} \leq \cos x \leq \frac{\sqrt{3}}{2}$, so $\frac{\sqrt{2}}{2}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \leq \int_{\pi/6}^{\pi/4} \cos x\,dx \leq \frac{\sqrt{3}}{2}\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$ [Property 8]; that is, $\frac{\sqrt{2}\,\pi}{24} \leq \int_{\pi/6}^{\pi/4} \cos x\,dx \leq \frac{\sqrt{3}\,\pi}{24}$.

**55.** If $1 \le x \le 4$, then $1 \le \sqrt{x} \le 2$, so $1(4-1) \le \int_1^4 \sqrt{x}\,dx \le 2(4-1)$; that is, $3 \le \int_1^4 \sqrt{x}\,dx \le 6$.

**56.** If $0 \le x \le 2$, then $1 \le 1 + x^2 \le 5$ and $\frac{1}{5} \le \frac{1}{1+x^2} \le 1$, so $\frac{1}{5}(2-0) \le \int_0^2 \frac{1}{1+x^2}\,dx \le 1(2-0)$;

that is, $\frac{2}{5} \le \int_0^2 \frac{1}{1+x^2}\,dx \le 2$.

**57.** If $\frac{\pi}{4} \le x \le \frac{\pi}{3}$, then $1 \le \tan x \le \sqrt{3}$, so $1\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ or $\frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \frac{\pi}{12}\sqrt{3}$.

**58.** Let $f(x) = x^3 - 3x + 3$ for $0 \le x \le 2$. Then $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$, so $f$ is decreasing on $(0,1)$ and increasing on $(1,2)$. $f$ has the absolute minimum value $f(1) = 1$. Since $f(0) = 3$ and $f(2) = 5$, the absolute maximum value of $f$ is $f(2) = 5$. Thus, $1 \le x^3 - 3x + 3 \le 5$ for $x$ in $[0,2]$. It follows from Property 8 that $1 \cdot (2-0) \le \int_0^2 \left(x^3 - 3x + 3\right) dx \le 5 \cdot (2-0)$; that is, $2 \le \int_0^2 \left(x^3 - 3x + 3\right) dx \le 10$.

**59.** The only critical number of $f(x) = xe^{-x}$ on $[0,2]$ is $x = 1$. Since $f(0) = 0$, $f(1) = e^{-1} \approx 0.368$, and $f(2) = 2e^{-2} \approx 0.271$, we know that the absolute minimum value of $f$ on $[0,2]$ is 0, and the absolute maximum is $e^{-1}$. By Property 8, $0 \le xe^{-x} \le e^{-1}$ for $0 \le x \le 2 \;\Rightarrow\; 0(2-0) \le \int_0^2 xe^{-x}\,dx \le e^{-1}(2-0) \;\Rightarrow\; 0 \le \int_0^2 xe^{-x}\,dx \le 2/e$.

**60.** Let $f(x) = x - 2\sin x$ for $\pi \le x \le 2\pi$. Then $f'(x) = 1 - 2\cos x$ and $f'(x) = 0 \;\Rightarrow\; \cos x = \frac{1}{2} \;\Rightarrow\; x = \frac{5\pi}{3}$. $f$ has the absolute maximum value $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} \approx 6.97$ since $f(\pi) = \pi$ and $f(2\pi) = 2\pi$ are both smaller than 6.97. Thus, $\pi \le f(x) \le \frac{5\pi}{3} + \sqrt{3} \;\Rightarrow\; \pi(2\pi - \pi) \le \int_\pi^{2\pi} f(x)\,dx \le \left(\frac{5\pi}{3} + \sqrt{3}\right)(2\pi - \pi)$; that is, $\pi^2 \le \int_\pi^{2\pi} (x - 2\sin x)\,dx \le \frac{5}{3}\pi^2 + \sqrt{3}\,\pi$.

**61.** $\sqrt{x^4+1} \ge \sqrt{x^4} = x^2$, so $\int_1^3 \sqrt{x^4+1}\,dx \ge \int_1^3 x^2\,dx = \frac{1}{3}\left(3^3 - 1^3\right) = \frac{26}{3}$.

**62.** $0 \le \sin x \le 1$ for $0 \le x \le \frac{\pi}{2}$, so $x \sin x \le x \;\Rightarrow\; \int_0^{\pi/2} x\sin x\,dx \le \int_0^{\pi/2} x\,dx = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - 0^2\right] = \frac{\pi^2}{8}$.

**63.** Using right endpoints as in the proof of Property 2, we calculate

$$\int_a^b cf(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n cf(x_i)\,\Delta x = \lim_{n\to\infty} c \sum_{i=1}^n f(x_i)\,\Delta x = c \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\,\Delta x = c\int_a^b f(x)\,dx.$$

**64.** As in the proof of Property 2, we write $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\,\Delta x$. Now $f(x_i) \ge 0$ and $\Delta x \ge 0$, so $f(x_i)\,\Delta x \ge 0$ and therefore $\sum_{i=1}^n f(x_i)\,\Delta x \ge 0$. But the limit of nonnegative quantities is nonnegative, so $\int_a^b f(x)\,dx \ge 0$.

**65.** Since $-|f(x)| \le f(x) \le |f(x)|$, it follows from Property 7 that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \;\Rightarrow\; \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Note that the definite integral is a real number, and so the following property applies: $-a \le b \le a \;\Rightarrow\; |b| \le a$ for all real numbers $b$ and nonnegative numbers $a$.

**66.** $\left|\int_0^{2\pi} f(x)\sin 2x\,dx\right| \le \int_0^{2\pi} |f(x)\sin 2x|\,dx$ [by Exercise 65] $= \int_0^{2\pi} |f(x)|\,|\sin 2x|\,dx \le \int_0^{2\pi} |f(x)|\,dx$ by Property 7, since $|\sin 2x| \le 1 \;\Rightarrow\; |f(x)|\,|\sin 2x| \le |f(x)|$.

**67.** To show that $f$ is integrable on $[0,1]$, we must show that $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\,\Delta x$ exists. Let $n$ denote a positive integer and divide the interval $[0,1]$ into $n$ equal subintervals $\left[0,\frac{1}{n}\right], \left[\frac{1}{n},\frac{2}{n}\right], \ldots, \left[\frac{n-1}{n},1\right]$. If we choose $x_i^*$ to be a rational number in the $i$th subinterval, then we obtain the Riemann sum $\sum\limits_{i=1}^{n} f(x_i^*)\cdot\frac{1}{n} = 0$, so $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\cdot\frac{1}{n} = \lim\limits_{n\to\infty} 0 = 0$. Now suppose we choose $x_i^*$ to be an irrational number. Then we get $\sum\limits_{i=1}^{n} f(x_i^*)\cdot\frac{1}{n} = \sum\limits_{i=1}^{n} 1\cdot\frac{1}{n} = n\cdot\frac{1}{n} = 1$ for each $n$, so $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\cdot\frac{1}{n} = \lim\limits_{n\to\infty} 1 = 1$. Since the value of $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\,\Delta x$ depends on the choice of the sample points $x_i^*$, the limit does not exist, and $f$ is not integrable on $[0,1]$.

**68.** Partition the interval $[0,1]$ into $n$ equal subintervals and choose $x_1^* = \frac{1}{n^2}$. Then with $f(x) = \frac{1}{x}$, $\sum\limits_{i=1}^{n} f(x_i^*)\,\Delta x \ge f(x_1^*)\Delta x = \frac{1}{1/n^2}\cdot\frac{1}{n} = n$. Thus, $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\,\Delta x$ can be made arbitrarily large and hence, $f$ is not integrable on $[0,1]$.

**69.** $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{i^4}{n^5} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{i^4}{n^4}\cdot\frac{1}{n} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\frac{i}{n}\right)^4\frac{1}{n}$. At this point, we need to recognize the limit as being of the form $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i)\,\Delta x$, where $\Delta x = (1-0)/n = 1/n$, $x_i = 0 + i\,\Delta x = i/n$, and $f(x) = x^4$. Thus, the definite integral is $\int_0^1 x^4\,dx$.

**70.** $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\frac{1}{1+(i/n)^2} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{1}{1+(i/n)^2}\cdot\frac{1}{n} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i)\,\Delta x$, where $\Delta x = (1-0)/n = 1/n$, $x_i = 0 + i\,\Delta x = i/n$, and $f(x) = \frac{1}{1+x^2}$. Thus, the definite integral is $\displaystyle\int_0^1 \frac{dx}{1+x^2}$.

**71.** Choose $x_i = 1 + \frac{i}{n}$ and $x_i^* = \sqrt{x_{i-1}x_i} = \sqrt{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)}$. Then

$$\begin{aligned}
\int_1^2 x^{-2}\,dx &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)} = \lim_{n\to\infty} n\sum_{i=1}^{n}\frac{1}{(n+i-1)(n+i)} \\
&= \lim_{n\to\infty} n\sum_{i=1}^{n}\left(\frac{1}{n+i-1}-\frac{1}{n+i}\right) \quad \text{[by the hint]} \quad = \lim_{n\to\infty} n\left(\sum_{i=0}^{n-1}\frac{1}{n+i}-\sum_{i=1}^{n}\frac{1}{n+i}\right) \\
&= \lim_{n\to\infty} n\left(\left[\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}\right]-\left[\frac{1}{n+1}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}\right]\right) \\
&= \lim_{n\to\infty} n\left(\frac{1}{n}-\frac{1}{2n}\right) = \lim_{n\to\infty}\left(1-\tfrac{1}{2}\right) = \tfrac{1}{2}
\end{aligned}$$