5.3 The Fundamental Theorem of Calculus

1. One process undoes what the other one does. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it on page 387.

2. (a) $g(x) = \int_0^x f(t)\,dt$, so $g(0) = \int_0^0 f(t)\,dt = 0$.

$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2} \cdot 1 \cdot 1$ [area of triangle] $= \frac{1}{2}$.

$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$ [below the x-axis]

$= \frac{1}{2} - \frac{1}{2} \cdot 1 \cdot 1 = 0$.

$g(3) = g(2) + \int_2^3 f(t)\,dt = 0 - \frac{1}{2} \cdot 1 \cdot 1 = -\frac{1}{2}$.

$g(4) = g(3) + \int_3^4 f(t)\,dt = -\frac{1}{2} + \frac{1}{2} \cdot 1 \cdot 1 = 0$.

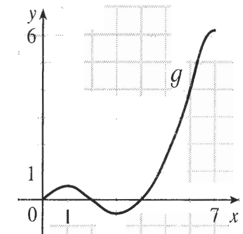
$g(5) = g(4) + \int_4^5 f(t)\,dt = 0 + 1.5 = 1.5$.

$g(6) = g(5) + \int_5^6 f(t)\,dt = 1.5 + 2.5 = 4$.

(b) $g(7) = g(6) + \int_6^7 f(t)\,dt \approx 4 + 2.2$ [estimate from the graph] $= 6.2$.

(c) The answers from part (a) and part (b) indicate that g has a minimum at $x = 3$ and a maximum at $x = 7$. This makes sense from the graph of f since we are subtracting area on $1 < x < 3$ and adding area on $3 < x < 7$.

(d)



3. (a) $g(x) = \int_0^x f(t)\,dt$.

$g(0) = \int_0^0 f(t)\,dt = 0$

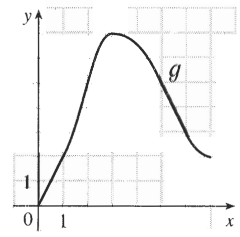
$g(1) = \int_0^1 f(t)\,dt = 1 \cdot 2 = 2$ [rectangle],

$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = g(1) + \int_1^2 f(t)\,dt$
$= 2 + 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 = 5$ [rectangle plus triangle],

$g(3) = \int_0^3 f(t)\,dt = g(2) + \int_2^3 f(t)\,dt = 5 + \frac{1}{2} \cdot 1 \cdot 4 = 7$,

$g(6) = g(3) + \int_3^6 f(t)\,dt$ [the integral is negative since f lies under the x-axis]
$= 7 + \left[-\left(\frac{1}{2} \cdot 2 \cdot 2 + 1 \cdot 2\right)\right] = 7 - 4 = 3$

(d)



(b) g is increasing on $(0, 3)$ because as x increases from 0 to 3, we keep adding more area.

(c) g has a maximum value when we start subtracting area; that is, at $x = 3$.

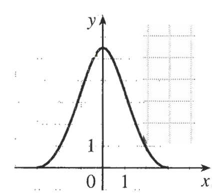
4. (a) $g(-3) = \int_{-3}^{-3} f(t)\,dt = 0$, $g(3) = \int_{-3}^{3} f(t)\,dt = \int_{-3}^{0} f(t)\,dt + \int_0^3 f(t)\,dt = 0$ by symmetry, since the area above the x-axis is the same as the area below the axis.

(b) From the graph, it appears that to the nearest $\frac{1}{2}$, $g(-2) = \int_{-3}^{-2} f(t)\,dt \approx 1$, $g(-1) = \int_{-3}^{-1} f(t)\,dt \approx 3\frac{1}{2}$, and $g(0) = \int_{-3}^{0} f(t)\,dt \approx 5\frac{1}{2}$.

(c) g is increasing on $(-3, 0)$ because as x increases from -3 to 0, we keep adding more area.

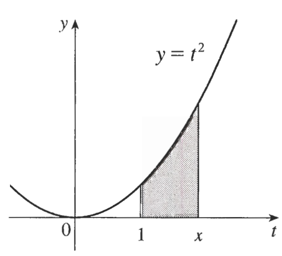
(d) g has a maximum value when we start subtracting area; that is, at $x = 0$.

(e)



(f) The graph of $g'(x)$ is the same as that of $f(x)$, as indicated by FTC1.

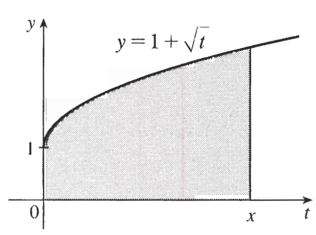
5.



(a) By FTC1 with $f(t) = t^2$ and $a = 1$, $g(x) = \int_1^x t^2\,dt \Rightarrow g'(x) = f(x) = x^2$.

(b) Using FTC2, $g(x) = \int_1^x t^2\,dt = \left[\frac{1}{3}t^3\right]_1^x = \frac{1}{3}x^3 - \frac{1}{3} \Rightarrow g'(x) = x^2$.

6.



(a) By FTC1 with $f(t) = 1 + \sqrt{t}$ and $a = 0$, $g(x) = \int_0^x \left(1 + \sqrt{t}\,\right) dt \Rightarrow g'(x) = f(x) = 1 + \sqrt{x}$.

(b) Using FTC2, $g(x) = \int_0^x \left(1 + \sqrt{t}\,\right) dt = \left[t + \frac{2}{3}t^{3/2}\right]_0^x = x + \frac{2}{3}x^{3/2} \Rightarrow g'(x) = 1 + x^{1/2} = 1 + \sqrt{x}$.

7. $f(t) = \dfrac{1}{t^3+1}$ and $g(x) = \displaystyle\int_1^x \frac{1}{t^3+1}\,dt$, so by FTC1, $g'(x) = f(x) = \dfrac{1}{x^3+1}$. Note that the lower limit, 1, could be any real number greater than -1 and not affect this answer.

8. $f(t) = e^{t^2-t}$ and $g(x) = \int_3^x e^{t^2-t}\,dt$, so by FTC1, $g'(x) = f(x) = e^{x^2-x}$.

9. $f(t) = t^2 \sin t$ and $g(y) = \int_2^y t^2 \sin t\,dt$, so by FTC1, $g'(y) = f(y) = y^2 \sin y$.

10. $f(x) = \sqrt{x^2+4}$ and $g(r) = \int_0^r \sqrt{x^2+4}\,dx$, so by FTC1, $g'(r) = f(r) = \sqrt{r^2+4}$.

11. $F(x) = \displaystyle\int_x^\pi \sqrt{1+\sec t}\,dt = -\int_\pi^x \sqrt{1+\sec t}\,dt \quad\Rightarrow\quad F'(x) = -\frac{d}{dx}\int_\pi^x \sqrt{1+\sec t}\,dt = -\sqrt{1+\sec x}$

12. $G(x) = \displaystyle\int_x^1 \cos\sqrt{t}\,dt = -\int_1^x \cos\sqrt{t}\,dt \quad\Rightarrow\quad G'(x) = -\frac{d}{dx}\int_1^x \cos\sqrt{t}\,dt = -\cos\sqrt{x}$

13. Let $u = \dfrac{1}{x}$. Then $\dfrac{du}{dx} = -\dfrac{1}{x^2}$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_2^{1/x} \arctan t\,dt = \frac{d}{du}\int_2^u \arctan t\,dt \cdot \frac{du}{dx} = \arctan u\,\frac{du}{dx} = -\frac{\arctan(1/x)}{x^2}.$$

14. Let $u = x^2$. Then $\dfrac{du}{dx} = 2x$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_0^{x^2} \sqrt{1+r^3}\,dr = \frac{d}{du}\int_0^u \sqrt{1+r^3}\,dr \cdot \frac{du}{dx} = \sqrt{1+u^3}(2x) = 2x\sqrt{1+(x^2)^3} = 2x\sqrt{1+x^6}.$$

15. Let $u = \tan x$. Then $\dfrac{du}{dx} = \sec^2 x$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_0^{\tan x} \sqrt{t+\sqrt{t}}\,dt = \frac{d}{du}\int_0^u \sqrt{t+\sqrt{t}}\,dt \cdot \frac{du}{dx} = \sqrt{u+\sqrt{u}}\,\frac{du}{dx} = \sqrt{\tan x + \sqrt{\tan x}}\,\sec^2 x.$$

16. Let $u = \cos x$. Then $\dfrac{du}{dx} = -\sin x$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_1^{\cos x} (1+v^2)^{10}\,dv = \frac{d}{du}\int_1^u (1+v^2)^{10}\,dv \cdot \frac{du}{dx} = (1+u^2)^{10}\frac{du}{dx} = -(1+\cos^2 x)^{10}\sin x.$$

17. Let $w = 1-3x$. Then $\dfrac{dw}{dx} = -3$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{dw}\dfrac{dw}{dx}$, so

$$y' = \frac{d}{dx}\int_{1-3x}^1 \frac{u^3}{1+u^2}\,du = \frac{d}{dw}\int_w^1 \frac{u^3}{1+u^2}\,du \cdot \frac{dw}{dx} = -\frac{d}{dw}\int_1^w \frac{u^3}{1+u^2}\,du \cdot \frac{dw}{dx} = -\frac{w^3}{1+w^2}(-3) = \frac{3(1-3x)^3}{1+(1-3x)^2}$$

18. Let $u = e^x$. Then $\dfrac{du}{dx} = e^x$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_{e^x}^0 \sin^3 t\,dt = \frac{d}{du}\int_u^0 \sin^3 t\,dt \cdot \frac{du}{dx} = -\frac{d}{du}\int_0^u \sin^3 t\,dt \cdot \frac{du}{dx} = -\sin^3 u \cdot e^x = -e^x\sin^3(e^x).$$

19. $\displaystyle\int_{-1}^2 (x^3-2x)\,dx = \left[\frac{x^4}{4} - x^2\right]_{-1}^2 = \left(\frac{2^4}{4} - 2^2\right) - \left(\frac{(-1)^4}{4} - (-1)^2\right) = (4-4) - \left(\tfrac{1}{4} - 1\right) = 0 - \left(-\tfrac{3}{4}\right) = \tfrac{3}{4}$

20. $\int_{-2}^5 6\,dx = \left[6x\right]_{-2}^5 = 6[5-(-2)] = 6(7) = 42$

21. $\int_1^4 (5 - 2t + 3t^2)\,dt = \left[5t - t^2 + t^3\right]_1^4 = (20 - 16 + 64) - (5 - 1 + 1) = 68 - 5 = 63$

22. $\int_0^1 \left(1 + \frac{1}{2}u^4 - \frac{2}{5}u^9\right) du = \left[u + \frac{1}{10}u^5 - \frac{1}{25}u^{10}\right]_0^1 = \left(1 + \frac{1}{10} - \frac{1}{25}\right) - 0 = \frac{53}{50}$

23. $\int_0^1 x^{4/5}\,dx = \left[\frac{5}{9}x^{9/5}\right]_0^1 = \frac{5}{9} - 0 = \frac{5}{9}$

24. $\int_1^8 \sqrt[3]{x}\,dx = \int_1^8 x^{1/3}\,dx = \left[\frac{3}{4}x^{4/3}\right]_1^8 = \frac{3}{4}(8^{4/3} - 1^{4/3}) = \frac{3}{4}(2^4 - 1) = \frac{3}{4}(16 - 1) = \frac{3}{4}(15) = \frac{45}{4}$

25. $\displaystyle\int_1^2 \frac{3}{t^4}\,dt = 3\int_1^2 t^{-4}\,dt = 3\left[\frac{t^{-3}}{-3}\right]_1^2 = \frac{3}{-3}\left[\frac{1}{t^3}\right]_1^2 = -1\left(\frac{1}{8} - 1\right) = \frac{7}{8}$

26. $\int_\pi^{2\pi} \cos\theta\,d\theta = \left[\sin\theta\right]_\pi^{2\pi} = \sin 2\pi - \sin\pi = 0 - 0 = 0$

27. $\int_0^2 x(2 + x^5)\,dx = \int_0^2 (2x + x^6)\,dx = \left[x^2 + \frac{1}{7}x^7\right]_0^2 = \left(4 + \frac{128}{7}\right) - (0 + 0) = \frac{156}{7}$

28. $\int_0^1 (3 + x\sqrt{x})\,dx = \int_0^1 (3 + x^{3/2})\,dx = \left[3x + \frac{2}{5}x^{5/2}\right]_0^1 = \left[\left(3 + \frac{2}{5}\right) - 0\right] = \frac{17}{5}$

29. $\displaystyle\int_1^9 \frac{x - 1}{\sqrt{x}}\,dx = \int_1^9 \left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right) dx = \int_1^9 (x^{1/2} - x^{-1/2})\,dx = \left[\frac{2}{3}x^{3/2} - 2x^{1/2}\right]_1^9$

$= \left(\frac{2}{3}\cdot 27 - 2\cdot 3\right) - \left(\frac{2}{3} - 2\right) = 12 - \left(-\frac{4}{3}\right) = \frac{40}{3}$

30. $\int_0^2 (y - 1)(2y + 1)\,dy = \int_0^2 (2y^2 - y - 1)\,dy = \left[\frac{2}{3}y^3 - \frac{1}{2}y^2 - y\right]_0^2 = \left(\frac{16}{3} - 2 - 2\right) - 0 = \frac{4}{3}$

31. $\int_0^{\pi/4} \sec^2 t\,dt = \left[\tan t\right]_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$

32. $\int_0^{\pi/4} \sec\theta\,\tan\theta\,d\theta = \left[\sec\theta\right]_0^{\pi/4} = \sec\frac{\pi}{4} - \sec 0 = \sqrt{2} - 1$

33. $\int_1^2 (1 + 2y)^2\,dy = \int_1^2 (1 + 4y + 4y^2)\,dy = \left[y + 2y^2 + \frac{4}{3}y^3\right]_1^2 = \left(2 + 8 + \frac{32}{3}\right) - \left(1 + 2 + \frac{4}{3}\right) = \frac{62}{3} - \frac{13}{3} = \frac{49}{3}$

34. $\int_0^1 \cosh t\,dt = \left[\sinh t\right]_0^1 = \sinh 1 - \sinh 0 = \sinh 1 \quad \left[\text{or } \frac{1}{2}(e - e^{-1})\right]$

35. $\displaystyle\int_1^9 \frac{1}{2x}\,dx = \frac{1}{2}\int_1^9 \frac{1}{x}\,dx = \frac{1}{2}\Big[\ln|x|\Big]_1^9 = \frac{1}{2}(\ln 9 - \ln 1) = \frac{1}{2}\ln 9 - 0 = \ln 9^{1/2} = \ln 3$

36. $\displaystyle\int_0^1 10^x\,dx = \left[\frac{10^x}{\ln 10}\right]_0^1 = \frac{10}{\ln 10} - \frac{1}{\ln 10} = \frac{9}{\ln 10}$

37. $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1 - t^2}}\,dt = 6\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1 - t^2}}\,dt = 6\left[\sin^{-1} t\right]_{1/2}^{\sqrt{3}/2} = 6\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right] = 6\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 6\left(\frac{\pi}{6}\right) = \pi$

38. $\displaystyle\int_0^1 \frac{4}{t^2 + 1}\,dt = 4\int_0^1 \frac{1}{1 + t^2}\,dt = 4\left[\tan^{-1} t\right]_0^1 = 4\left(\tan^{-1} 1 - \tan^{-1} 0\right) = 4\left(\frac{\pi}{4} - 0\right) = \pi$

39. $\int_{-1}^1 e^{u+1}\,du = \left[e^{u+1}\right]_{-1}^1 = e^2 - e^0 = e^2 - 1 \quad [\text{or start with } e^{u+1} = e^u e^1]$

40. $\displaystyle\int_1^2 \frac{4 + u^2}{u^3}\,du = \int_1^2 (4u^{-3} + u^{-1})\,du = \left[\frac{4}{-2}u^{-2} + \ln|u|\right]_1^2 = \left[\frac{-2}{u^2} + \ln u\right]_1^2 = \left(-\frac{1}{2} + \ln 2\right) - (-2 + \ln 1) = \frac{3}{2} + \ln 2$

41. If $f(x) = \begin{cases} \sin x & \text{if } 0 \le x < \pi/2 \\ \cos x & \text{if } \pi/2 \le x \le \pi \end{cases}$ then

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^{\pi} \cos x\,dx = \big[-\cos x\big]_0^{\pi/2} + [\sin x]_{\pi/2}^{\pi} = -\cos\tfrac{\pi}{2} + \cos 0 + \sin\pi - \sin\tfrac{\pi}{2}$$
$$= -0 + 1 + 0 - 1 = 0$$

Note that f is integrable by Theorem 3 in Section 5.2.

42. If $f(x) = \begin{cases} 2 & \text{if } -2 \le x \le 0 \\ 4 - x^2 & \text{if } 0 < x \le 2 \end{cases}$ then

$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{0} 2\,dx + \int_0^2 (4 - x^2)\,dx = \big[2x\big]_{-2}^{0} + \big[4x - \tfrac{1}{3}x^3\big]_0^2 = [0 - (-4)] + \big(\tfrac{16}{3} - 0\big) = \tfrac{28}{3}$$

Note that f is integrable by Theorem 3 in Section 5.2

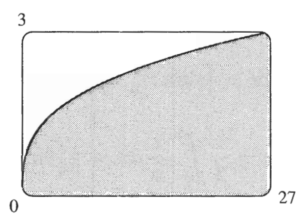
43. $f(x) = x^{-4}$ is not continuous on the interval $[-2, 1]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = 0$, so $\int_{-2}^{1} x^{-4}\,dx$ does not exist.

44. $f(x) = \dfrac{4}{x^3}$ is not continuous on the interval $[-1, 2]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = 0$, so $\displaystyle\int_{-1}^{2} \frac{4}{x^3}\,dx$ does not exist.

45. $f(\theta) = \sec\theta\,\tan\theta$ is not continuous on the interval $[\pi/3, \pi]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = \pi/2$, so $\int_{\pi/3}^{\pi} \sec\theta\,\tan\theta\,d\theta$ does not exist.

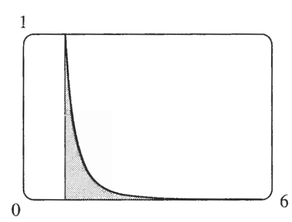
46. $f(x) = \sec^2 x$ is not continuous on the interval $[0, \pi]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = \pi/2$, so $\int_0^{\pi} \sec^2 x\,dx$ does not exist.

47. From the graph, it appears that the area is about 60. The actual area is $\int_0^{27} x^{1/3}dx = \left[\tfrac{3}{4}x^{4/3}\right]_0^{27} = \tfrac{3}{4}\cdot 81 - 0 = \tfrac{243}{4} = 60.75$. This is $\tfrac{3}{4}$ of the area of the viewing rectangle.

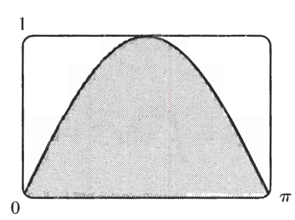


48. From the graph, it appears that the area is about $\frac{1}{3}$. The actual area is

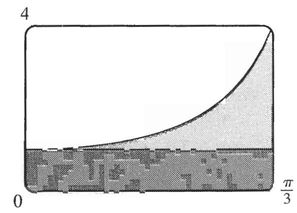
$$\int_1^6 x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_1^6 = \left[\frac{-1}{3x^3}\right]_1^6 = -\frac{1}{3\cdot 216} + \frac{1}{3} = \frac{215}{648} \approx 0.3318.\text{W}$$



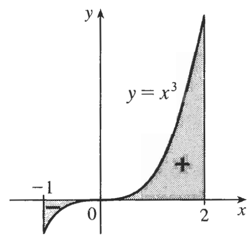
49. It appears that the area under the graph is about $\frac{2}{3}$ of the area of the viewing rectangle, or about $\frac{2}{3}\pi \approx 2.1$. The actual area is $\int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = (-\cos\pi) - (-\cos 0) = -(-1) + 1 = 2$.



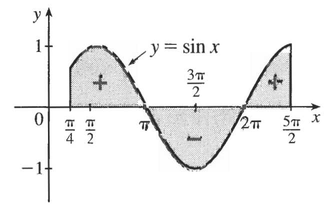
50. Splitting up the region as shown, we estimate that the area under the graph is $\frac{\pi}{3} + \frac{1}{4}\left(3 \cdot \frac{\pi}{3}\right) \approx 1.8$. The actual area is $\int_0^{\pi/3} \sec^2 x\,dx = [\tan x]_0^{\pi/3} = \sqrt{3} - 0 = \sqrt{3} \approx 1.73$.



51. $\int_{-1}^{2} x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^{2} = 4 - \frac{1}{4} = \frac{15}{4} = 3.75$



52. $\int_{\pi/4}^{5\pi/2} \sin x\,dx = [-\cos x]_{\pi/4}^{5\pi/2} = 0 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$



53. $g(x) = \displaystyle\int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du = \int_{2x}^{0} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du = -\int_{0}^{2x} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du \quad\Rightarrow$

$g'(x) = -\dfrac{(2x)^2-1}{(2x)^2+1} \cdot \dfrac{d}{dx}(2x) + \dfrac{(3x)^2-1}{(3x)^2+1} \cdot \dfrac{d}{dx}(3x) = -2 \cdot \dfrac{4x^2-1}{4x^2+1} + 3 \cdot \dfrac{9x^2-1}{9x^2+1}$

54. $g(x) = \displaystyle\int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt = \int_{\tan x}^{1} \frac{dt}{\sqrt{2+t^4}} + \int_{1}^{x^2} \frac{dt}{\sqrt{2+t^4}} = -\int_{1}^{\tan x} \frac{dt}{\sqrt{2+t^4}} + \int_{1}^{x^2} \frac{dt}{\sqrt{2+t^4}} \quad\Rightarrow$

$g'(x) = \dfrac{-1}{\sqrt{2+\tan^4 x}}\,\dfrac{d}{dx}(\tan x) + \dfrac{1}{\sqrt{2+x^8}}\,\dfrac{d}{dx}\left(x^2\right) = -\dfrac{\sec^2 x}{\sqrt{2+\tan^4 x}} + \dfrac{2x}{\sqrt{2+x^8}}$

55. $y = \int_{\sqrt{x}}^{x^3} \sqrt{t}\sin t\,dt = \int_{\sqrt{x}}^{1} \sqrt{t}\sin t\,dt + \int_{1}^{x^3} \sqrt{t}\sin t\,dt = -\int_{1}^{\sqrt{x}} \sqrt{t}\sin t\,dt + \int_{1}^{x^3} \sqrt{t}\sin t\,dt \quad\Rightarrow$

$y' = -\sqrt[4]{x}\left(\sin\sqrt{x}\right) \cdot \dfrac{d}{dx}\left(\sqrt{x}\right) + x^{3/2}\sin(x^3) \cdot \dfrac{d}{dx}\left(x^3\right) = -\dfrac{\sqrt[4]{x}\sin\sqrt{x}}{2\sqrt{x}} + x^{3/2}\sin(x^3)(3x^2)$

$= 3x^{7/2}\sin(x^3) - \dfrac{\sin\sqrt{x}}{2\sqrt[4]{x}}$

56. $y = \int_{\cos x}^{5x} \cos(u^2)\,du = \int_{0}^{5x} \cos(u^2)\,du - \int_{0}^{\cos x} \cos(u^2)\,du \quad\Rightarrow$

$y' = \cos(25x^2) \cdot \dfrac{d}{dx}(5x) - \cos(\cos^2 x) \cdot \dfrac{d}{dx}(\cos x) = \cos(25x^2) \cdot 5 - \cos(\cos^2 x) \cdot (-\sin x)$

$= 5\cos(25x^2) + \sin x\cos(\cos^2 x)$

57. $F(x) = \displaystyle\int_{1}^{x} f(t)\,dt \quad\Rightarrow\quad F'(x) = f(x) = \int_{1}^{x^2} \frac{\sqrt{1+u^4}}{u}\,du \quad \left[\text{since } f(t) = \int_{1}^{t^2} \frac{\sqrt{1+u^4}}{u}\,du\right] \quad\Rightarrow$

$F''(x) = f'(x) = \dfrac{\sqrt{1+(x^2)^4}}{x^2} \cdot \dfrac{d}{dx}\left(x^2\right) = \dfrac{\sqrt{1+x^8}}{x^2} \cdot 2x = \dfrac{2\sqrt{1+x^8}}{x}$. So $F''(2) = \sqrt{1+2^8} = \sqrt{257}$.

58. For the curve to be concave upward, we must have $y'' > 0$.

$y = \displaystyle\int_{0}^{x} \frac{1}{1+t+t^2}\,dt \quad\Rightarrow\quad y' = \frac{1}{1+x+x^2} \quad\Rightarrow\quad y'' = \frac{-(1+2x)}{(1+x+x^2)^2}$. For this expression to be positive, we must have $(1+2x) < 0$, since $(1+x+x^2)^2 > 0$ for all x. $(1+2x) < 0 \;\Leftrightarrow\; x < -\frac{1}{2}$. Thus, the curve is concave upward on $\left(-\infty, -\frac{1}{2}\right)$.

59. By FTC2, $\int_1^4 f'(x)\,dx = f(4) - f(1)$, so $17 = f(4) - 12 \Rightarrow f(4) = 17 + 12 = 29$.

60. (a) $\text{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\,dt \quad\Rightarrow\quad \int_0^x e^{-t^2}\,dt = \dfrac{\sqrt{\pi}}{2}\,\text{erf}(x)$. By Property 5 of definite integrals in Section 5.2,

$\int_0^b e^{-t^2}\,dt = \int_0^a e^{-t^2}\,dt + \int_a^b e^{-t^2}\,dt$, so

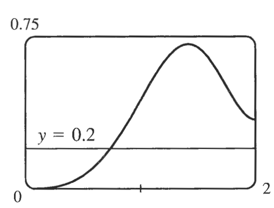
$$\int_a^b e^{-t^2}\,dt = \int_0^b e^{-t^2}\,dt - \int_0^a e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\,\text{erf}(b) - \frac{\sqrt{\pi}}{2}\,\text{erf}(a) = \tfrac{1}{2}\sqrt{\pi}\,[\text{erf}(b) - \text{erf}(a)].$$

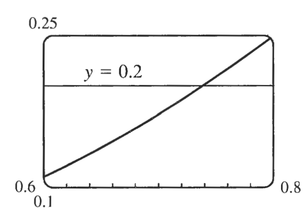
(b) $y = e^{x^2}\text{erf}(x) \quad\Rightarrow\quad y' = 2xe^{x^2}\text{erf}(x) + e^{x^2}\text{erf}'(x) = 2xy + e^{x^2}\cdot\dfrac{2}{\sqrt{\pi}}e^{-x^2}$ [by FTC1] $= 2xy + \dfrac{2}{\sqrt{\pi}}$.

61. (a) The Fresnel function $S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt$ has local maximum values where $0 = S'(x) = \sin\left(\frac{\pi}{2}t^2\right)$ and S' changes from positive to negative. For $x > 0$, this happens when $\frac{\pi}{2}x^2 = (2n-1)\pi$ [odd multiples of π] $\Leftrightarrow$ $x^2 = 2(2n-1) \Leftrightarrow x = \sqrt{4n-2}$, n any positive integer. For $x < 0$, S' changes from positive to negative where $\frac{\pi}{2}x^2 = 2n\pi$ [even multiples of π] $\Leftrightarrow x^2 = 4n \Leftrightarrow x = -2\sqrt{n}$. S' does not change sign at $x = 0$.

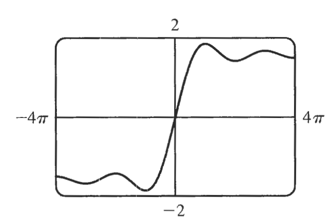
(b) S is concave upward on those intervals where $S''(x) > 0$. Differentiating our expression for $S'(x)$, we get $S''(x) = \cos\left(\frac{\pi}{2}x^2\right)\left(2\frac{\pi}{2}x\right) = \pi x\cos\left(\frac{\pi}{2}x^2\right)$. For $x > 0$, $S''(x) > 0$ where $\cos(\frac{\pi}{2}x^2) > 0 \Leftrightarrow 0 < \frac{\pi}{2}x^2 < \frac{\pi}{2}$ or $\left(2n - \frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n + \frac{1}{2}\right)\pi$, n any integer $\Leftrightarrow 0 < x < 1$ or $\sqrt{4n-1} < x < \sqrt{4n+1}$, n any positive integer. For $x < 0$, $S''(x) > 0$ where $\cos(\frac{\pi}{2}x^2) < 0 \Leftrightarrow \left(2n - \frac{3}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n - \frac{1}{2}\right)\pi$, n any integer $\Leftrightarrow$ $4n - 3 < x^2 < 4n - 1 \Leftrightarrow \sqrt{4n-3} < |x| < \sqrt{4n-1} \Rightarrow \sqrt{4n-3} < -x < \sqrt{4n-1} \Rightarrow$ $-\sqrt{4n-3} > x > -\sqrt{4n-1}$, so the intervals of upward concavity for $x < 0$ are $\left(-\sqrt{4n-1}, -\sqrt{4n-3}\right)$, n any positive integer. To summarize: S is concave upward on the intervals $(0,1)$, $\left(-\sqrt{3}, -1\right)$, $\left(\sqrt{3}, \sqrt{5}\right)$, $\left(-\sqrt{7}, -\sqrt{5}\right)$, $\left(\sqrt{7}, 3\right), \ldots$.

(c) In Maple, we use `plot({int(sin(Pi*t^2/2),t=0..x),0.2},x=0..2);`. Note that Maple recognizes the Fresnel function, calling it `FresnelS(x)`. In Mathematica, we use `Plot[{Integrate[Sin[Pi*t^2/2],{t,0,x}],0.2},{x,0,2}]`. In Derive, we load the utility file `FRESNEL` and plot `FRESNEL_SIN(x)`. From the graphs, we see that $\int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt = 0.2$ at $x \approx 0.74$.





62. (a) In Maple, we should start by setting `si:=int(sin(t)/t,t=0..x);`. In Mathematica, the command is `si=Integrate[Sin[t]/t,{t,0,x}]`. Note that both systems recognize this function; Maple calls it `Si(x)` and Mathematica calls it `SinIntegral[x]`. In Maple, the command to generate the graph is `plot(si,x=-4*Pi..4*Pi);`. In Mathematica, it is `Plot[si,{x,-4*Pi,4*Pi}]`. In Derive, we load the utility file `EXP_INT` and plot `SI(x)`.



(b) $\text{Si}(x)$ has local maximum values where $\text{Si}'(x)$ changes from positive to negative, passing through 0. From the Fundamental Theorem we know that $\text{Si}'(x) = \frac{d}{dx}\int_0^x \frac{\sin t}{t}\,dt = \frac{\sin x}{x}$, so we must have $\sin x = 0$ for a maximum, and for $x > 0$ we must have $x = (2n-1)\pi$, n any positive integer, for Si' to be changing from positive to negative at x. For $x < 0$, we must have $x = 2n\pi$, n any positive integer, for a maximum, since the denominator of $\text{Si}'(x)$ is negative for $x < 0$. Thus, the local maxima occur at $x = \pi, -2\pi, 3\pi, -4\pi, 5\pi, -6\pi, \ldots$.

(c) To find the first inflection point, we solve $\text{Si}''(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} = 0$. We can see from the graph that the first inflection point lies somewhere between $x = 3$ and $x = 5$. Using a root finder gives the value $x \approx 4.4934$. To find the y-coordinate of the inflection point, we evaluate $\text{Si}(4.4934) \approx 1.6556$. So the coordinates of the first inflection point to the right of the origin are about $(4.4934, 1.6556)$. Alternatively, we could graph $S''(x)$ and estimate the first positive x-value at which it changes sign.

(d) It seems from the graph that the function has horizontal asymptotes at $y \approx 1.5$, with $\lim_{x\to\pm\infty} \text{Si}(x) \approx \pm 1.5$ respectively. Using the limit command, we get $\lim_{x\to\infty} \text{Si}(x) = \frac{\pi}{2}$. Since $\text{Si}(x)$ is an odd function, $\lim_{x\to-\infty} \text{Si}(x) = -\frac{\pi}{2}$. So $\text{Si}(x)$ has the horizontal asymptotes $y = \pm\frac{\pi}{2}$.

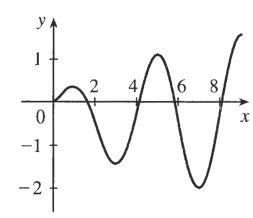
(e) We use the `fsolve` command in Maple (or `FindRoot` in Mathematica) to find that the solution is $x \approx 1.1$. Or, as in Exercise 61(c), we graph $y = \text{Si}(x)$ and $y = 1$ on the same screen to see where they intersect.

63. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 1, 3, 5, 7$, and 9. g has local maxima at $x = 1$ and 5 (since $f = g'$ changes from positive to negative there) and local minima at $x = 3$ and 7. There is no local maximum or minimum at $x = 9$, since f is not defined for $x > 9$.

(b) We can see from the graph that $\left|\int_0^1 f\,dt\right| < \left|\int_1^3 f\,dt\right| < \left|\int_3^5 f\,dt\right| < \left|\int_5^7 f\,dt\right| < \left|\int_7^9 f\,dt\right|$. So $g(1) = \left|\int_0^1 f\,dt\right|$, $g(5) = \int_0^5 f\,dt = g(1) - \left|\int_1^3 f\,dt\right| + \left|\int_3^5 f\,dt\right|$, and $g(9) = \int_0^9 f\,dt = g(5) - \left|\int_5^7 f\,dt\right| + \left|\int_7^9 f\,dt\right|$. Thus, $g(1) < g(5) < g(9)$, and so the absolute maximum of $g(x)$ occurs at $x = 9$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on (approximately) $\left(\frac{1}{2}, 2\right)$, $(4, 6)$ and $(8, 9)$. So g is concave downward on these intervals.

(d)

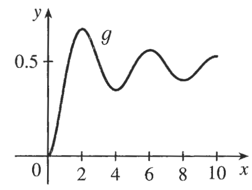


64. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 2, 4, 6, 8$, and 10. g has local maxima at $x = 2$ and 6 (since $f = g'$ changes from positive to negative there) and local minima at $x = 4$ and 8. There is no local maximum or minimum at $x = 10$, since f is not defined for $x > 10$.

(b) We can see from the graph that $\left|\int_0^2 f\,dt\right| > \left|\int_2^4 f\,dt\right| > \left|\int_4^6 f\,dt\right| > \left|\int_6^8 f\,dt\right| > \left|\int_8^{10} f\,dt\right|$. So $g(2) = \left|\int_0^2 f\,dt\right|$, $g(6) = \int_0^6 f\,dt = g(2) - \left|\int_2^4 f\,dt\right| + \left|\int_4^6 f\,dt\right|$, and $g(10) = \int_0^{10} f\,dt = g(6) - \left|\int_6^8 f\,dt\right| + \left|\int_8^{10} f\,dt\right|$. Thus, $g(2) > g(6) > g(10)$, and so the absolute maximum of $g(x)$ occurs at $x = 2$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on $(1, 3)$, $(5, 7)$ and $(9, 10)$. So g is concave downward on these intervals.

(d)



65. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4} = \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

66. $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right) = \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} \sqrt{\frac{i}{n}} = \int_0^1 \sqrt{x}\,dx = \left[\frac{2x^{3/2}}{3}\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$

67. Suppose $h < 0$. Since f is continuous on $[x+h, x]$, the Extreme Value Theorem says that there are numbers u and v in $[x+h, x]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x+h, x]$. By Property 8 of integrals, $m(-h) \le \int_{x+h}^{x} f(t)\,dt \le M(-h)$; that is, $f(u)(-h) \le -\int_x^{x+h} f(t)\,dt \le f(v)(-h)$.

Since $-h > 0$, we can divide this inequality by $-h$: $f(u) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(v)$. By Equation 2, $\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$ for $h \ne 0$, and hence $f(u) \le \frac{g(x+h)-g(x)}{h} \le f(v)$, which is Equation 3 in the case where $h < 0$.

68. $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = \frac{d}{dx}\left[\int_{g(x)}^{a} f(t)\,dt + \int_a^{h(x)} f(t)\,dt\right]$ [where a is in the domain of f]

$$= \frac{d}{dx}\left[-\int_a^{g(x)} f(t)\,dt\right] + \frac{d}{dx}\left[\int_a^{h(x)} f(t)\,dt\right] = -f(g(x))\,g'(x) + f(h(x))\,h'(x)$$

$$= f(h(x))\,h'(x) - f(g(x))\,g'(x)$$

69. (a) Let $f(x) = \sqrt{x} \Rightarrow f'(x) = 1/(2\sqrt{x}) > 0$ for $x > 0 \Rightarrow f$ is increasing on $(0, \infty)$. If $x \ge 0$, then $x^3 \ge 0$, so $1 + x^3 \ge 1$ and since f is increasing, this means that $f(1+x^3) \ge f(1) \Rightarrow \sqrt{1+x^3} \ge 1$ for $x \ge 0$. Next let $g(t) = t^2 - t \Rightarrow g'(t) = 2t - 1 \Rightarrow g'(t) > 0$ when $t \ge 1$. Thus, g is increasing on $(1, \infty)$. And since $g(1) = 0$, $g(t) \ge 0$ when $t \ge 1$. Now let $t = \sqrt{1+x^3}$, where $x \ge 0$. $\sqrt{1+x^3} \ge 1$ (from above) $\Rightarrow t \ge 1 \Rightarrow g(t) \ge 0 \Rightarrow (1+x^3) - \sqrt{1+x^3} \ge 0$ for $x \ge 0$. Therefore, $1 \le \sqrt{1+x^3} \le 1 + x^3$ for $x \ge 0$.

(b) From part (a) and Property 7: $\int_0^1 1\,dx \le \int_0^1 \sqrt{1+x^3}\,dx \le \int_0^1 (1+x^3)\,dx \Leftrightarrow$ $[x]_0^1 \le \int_0^1 \sqrt{1+x^3}\,dx \le \left[x + \frac{1}{4}x^4\right]_0^1 \Leftrightarrow 1 \le \int_0^1 \sqrt{1+x^3}\,dx \le 1 + \frac{1}{4} = 1.25$.

70. (a) For $0 \le x \le 1$, we have $x^2 \le x$. Since $f(x) = \cos x$ is a decreasing function on $[0, 1]$, $\cos(x^2) \ge \cos x$.

(b) $\pi/6 < 1$, so by part (a), $\cos(x^2) \ge \cos x$ on $[0, \pi/6]$. Thus,

$\int_0^{\pi/6} \cos(x^2)\,dx \ge \int_0^{\pi/6} \cos x\,dx = [\sin x]_0^{\pi/6} = \sin(\pi/6) - \sin 0 = \frac{1}{2} - 0 = \frac{1}{2}$.

71. $0 < \dfrac{x^2}{x^4 + x^2 + 1} < \dfrac{x^2}{x^4} = \dfrac{1}{x^2}$ on $[5, 10]$, so

$$0 \le \int_5^{10} \frac{x^2}{x^4 + x^2 + 1}\,dx < \int_5^{10} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_5^{10} = -\frac{1}{10} - \left(-\frac{1}{5}\right) = \frac{1}{10} = 0.1.$$

72. (a) If $x < 0$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x 0\,dt = 0$.

If $0 \le x \le 1$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt = \left[\frac{1}{2}t^2\right]_0^x = \frac{1}{2}x^2$.

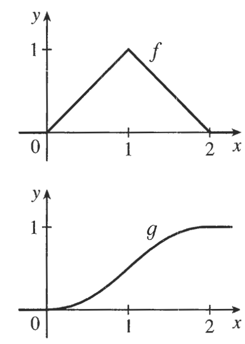
If $1 < x \le 2$, then

$$g(x) = \int_0^x f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt = g(1) + \int_1^x (2 - t)\,dt$$
$$= \tfrac{1}{2}(1)^2 + \left[2t - \tfrac{1}{2}t^2\right]_1^x = \tfrac{1}{2} + \left(2x - \tfrac{1}{2}x^2\right) - \left(2 - \tfrac{1}{2}\right) = 2x - \tfrac{1}{2}x^2 - 1.$$

If $x > 2$, then $g(x) = \int_0^x f(t)\,dt = g(2) + \int_2^x 0\,dt = 1 + 0 = 1$. So

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}x^2 & \text{if } 0 \le x \le 1 \\ 2x - \frac{1}{2}x^2 - 1 & \text{if } 1 < x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$$

(b)



(c) f is not differentiable at its corners at $x = 0$, 1, and 2. f is differentiable on $(-\infty, 0)$, $(0, 1)$, $(1, 2)$ and $(2, \infty)$.
g is differentiable on $(-\infty, \infty)$.

73. Using FTC1, we differentiate both sides of $6 + \displaystyle\int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$ to get $\dfrac{f(x)}{x^2} = 2\dfrac{1}{2\sqrt{x}} \Rightarrow f(x) = x^{3/2}$.

To find a, we substitute $x = a$ in the original equation to obtain $6 + \displaystyle\int_a^a \frac{f(t)}{t^2}\,dt = 2\sqrt{a} \Rightarrow 6 + 0 = 2\sqrt{a} \Rightarrow$
$3 = \sqrt{a} \Rightarrow a = 9$.

74. $B = 3A \Rightarrow \int_0^b e^x\,dx = 3\int_0^a e^x\,dx \Rightarrow [e^x]_0^b = 3\,[e^x]_0^a \Rightarrow e^b - 1 = 3(e^a - 1) \Rightarrow e^b = 3e^a - 2 \Rightarrow$
$b = \ln(3e^a - 2)$

75. (a) Let $F(t) = \int_0^t f(s)\,ds$. Then, by FTC1, $F'(t) = f(t) =$ rate of depreciation, so $F(t)$ represents the loss in value over the interval $[0, t]$.

(b) $C(t) = \dfrac{1}{t}\left[A + \displaystyle\int_0^t f(s)\,ds\right] = \dfrac{A + F(t)}{t}$ represents the average expenditure per unit of t during the interval $[0, t]$, assuming that there has been only one overhaul during that time period. The company wants to minimize average expenditure.

(c) $C(t) = \dfrac{1}{t}\left[A + \displaystyle\int_0^t f(s)\,ds\right]$. Using FTC1, we have $C'(t) = -\dfrac{1}{t^2}\left[A + \displaystyle\int_0^t f(s)\,ds\right] + \dfrac{1}{t}f(t)$.

$$C'(t) = 0 \Rightarrow t\,f(t) = A + \int_0^t f(s)\,ds \Rightarrow f(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = C(t).$$

76. (a) $C(t) = \frac{1}{t}\int_0^t [f(s)+g(s)]\,ds$. Using FTC1 and the Product Rule, we have

$C'(t) = \frac{1}{t}[f(t)+g(t)] - \frac{1}{t^2}\int_0^t [f(s)+g(s)]\,ds$. Set $C'(t)=0$: $\frac{1}{t}[f(t)+g(t)] - \frac{1}{t^2}\int_0^t [f(s)+g(s)]\,ds = 0 \Rightarrow$

$[f(t)+g(t)] - \frac{1}{t}\int_0^t [f(s)+g(s)]\,ds = 0 \Rightarrow [f(t)+g(t)] - C(t) = 0 \Rightarrow C(t) = f(t)+g(t)$.

(b) For $0 \le t \le 30$, we have $D(t) = \int_0^t \left(\frac{V}{15} - \frac{V}{450}s\right) ds = \left[\frac{V}{15}s - \frac{V}{900}s^2\right]_0^t = \frac{V}{15}t - \frac{V}{900}t^2$.

So $D(t) = V \Rightarrow \frac{V}{15}t - \frac{V}{900}t^2 = V \Rightarrow 60t - t^2 = 900 \Rightarrow t^2 - 60t + 900 = 0 \Rightarrow$

$(t-30)^2 = 0 \Rightarrow t = 30$. So the length of time T is 30 months.

(c) $C(t) = \frac{1}{t}\int_0^t \left(\frac{V}{15} - \frac{V}{450}s + \frac{V}{12{,}900}s^2\right) ds = \frac{1}{t}\left[\frac{V}{15}s - \frac{V}{900}s^2 + \frac{V}{38{,}700}s^3\right]_0^t$

$= \frac{1}{t}\left(\frac{V}{15}t - \frac{V}{900}t^2 + \frac{V}{38{,}700}t^3\right) = \frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2 \Rightarrow$

$C'(t) = -\frac{V}{900} + \frac{V}{19{,}350}t = 0$ when $\frac{1}{19{,}350}t = \frac{1}{900} \Rightarrow t = 21.5$.

$C(21.5) = \frac{V}{15} - \frac{V}{900}(21.5) + \frac{V}{38{,}700}(21.5)^2 \approx 0.05472V$, $C(0) = \frac{V}{15} \approx 0.06667V$, and

$C(30) = \frac{V}{15} - \frac{V}{900}(30) + \frac{V}{38{,}700}(30)^2 \approx 0.05659V$, so the absolute minimum is $C(21.5) \approx 0.05472V$.

(d) As in part (c), we have $C(t) = \frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2$, so $C(t) = f(t)+g(t) \Leftrightarrow$

$\frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2 = \frac{V}{15} - \frac{V}{450}t + \frac{V}{12{,}900}t^2 \Leftrightarrow$

$t^2\left(\frac{1}{12{,}900} - \frac{1}{38{,}700}\right) = t\left(\frac{1}{450} - \frac{1}{900}\right) \Leftrightarrow t = \frac{1/900}{2/38{,}700} = \frac{43}{2} = 21.5$.

This is the value of t that we obtained as the critical number of C in part (c), so we have verified the result of (a) in this case.