7 ☐ TECHNIQUES OF INTEGRATION

7.1 Integration by Parts

1. Let $u = \ln x$, $dv = x^2\,dx \;\Rightarrow\; du = \frac{1}{x}\,dx$, $v = \frac{1}{3}x^3$. Then by Equation 2,

$$\int x^2 \ln x\,dx = (\ln x)\left(\tfrac{1}{3}x^3\right) - \int \left(\tfrac{1}{3}x^3\right)\left(\tfrac{1}{x}\right)dx = \tfrac{1}{3}x^3\,\ln x - \tfrac{1}{3}\int x^2\,dx = \tfrac{1}{3}x^3\,\ln x - \tfrac{1}{3}\left(\tfrac{1}{3}x^3\right) + C$$
$$= \tfrac{1}{3}x^3\,\ln x - \tfrac{1}{9}x^3 + C \quad \left[\text{or } \tfrac{1}{3}x^3\left(\ln x - \tfrac{1}{3}\right) + C\right]$$

2. Let $u = \theta$, $dv = \cos\theta\,d\theta \;\Rightarrow\; du = d\theta$, $v = \sin\theta$. Then by Equation 2,

$$\int \theta\cos\theta\,d\theta = \theta\sin\theta - \int \sin\theta\,d\theta = \theta\sin\theta + \cos\theta + C.$$

Note: A mnemonic device which is helpful for selecting u when using integration by parts is the LIATE principle of precedence for u:

Logarithmic
Inverse trigonometric
Algebraic
Trigonometric
Exponential

If the integrand has several factors, then we try to choose among them a u which appears as high as possible on the list. For example, in $\int xe^{2x}\,dx$ the integrand is xe^{2x}, which is the product of an algebraic function (x) and an exponential function (e^{2x}). Since Algebraic appears before Exponential, we choose $u = x$. Sometimes the integration turns out to be similar regardless of the selection of u and dv, but it is advisable to refer to LIATE when in doubt.

3. Let $u = x$, $dv = \cos 5x\,dx \;\Rightarrow\; du = dx$, $v = \frac{1}{5}\sin 5x$. Then by Equation 2,

$$\int x\cos 5x\,dx = \tfrac{1}{5}x\sin 5x - \int \tfrac{1}{5}\sin 5x\,dx = \tfrac{1}{5}x\sin 5x + \tfrac{1}{25}\cos 5x + C.$$

4. Let $u = x$, $dv = e^{-x}\,dx \;\Rightarrow\; du = dx$, $v = -e^{-x}$. Then $\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$.

5. Let $u = r$, $dv = e^{r/2}\,dr \;\Rightarrow\; du = dr$, $v = 2e^{r/2}$. Then $\int re^{r/2}\,dr = 2re^{r/2} - \int 2e^{r/2}\,dr = 2re^{r/2} - 4e^{r/2} + C$.

6. Let $u = t$, $dv = \sin 2t\,dt \;\Rightarrow\; du = dt$, $v = -\frac{1}{2}\cos 2t$. Then

$$\int t\sin 2t\,dt = -\tfrac{1}{2}t\cos 2t + \tfrac{1}{2}\int\cos 2t\,dt = -\tfrac{1}{2}t\cos 2t + \tfrac{1}{4}\sin 2t + C.$$

7. Let $u = x^2$, $dv = \sin\pi x\,dx \;\Rightarrow\; du = 2x\,dx$ and $v = -\frac{1}{\pi}\cos\pi x$. Then

$I = \int x^2\,\sin\pi x\,dx = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi}\int x\cos\pi x\,dx$ $(\star)$. Next let $U = x$, $dV = \cos\pi x\,dx \;\Rightarrow\; dU = dx$,

$V = \frac{1}{\pi}\sin\pi x$, so $\int x\cos\pi x\,dx = \frac{1}{\pi}x\sin\pi x - \frac{1}{\pi}\int\sin\pi x\,dx = \frac{1}{\pi}x\sin\pi x + \frac{1}{\pi^2}\cos\pi x + C_1$.

Substituting for $\int x\cos\pi x\,dx$ in $(\star)$, we get

$I = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi}\left(\frac{1}{\pi}x\sin\pi x + \frac{1}{\pi^2}\cos\pi x + C_1\right) = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi^2}x\sin\pi x + \frac{2}{\pi^3}\cos\pi x + C$, where $C = \frac{2}{\pi}C_1$.

8. Let $u = x^2$, $dv = \cos mx\,dx \Rightarrow du = 2x\,dx$, $v = \dfrac{1}{m}\sin mx$. Then

$I = \int x^2 \cos mx\,dx = \dfrac{1}{m}x^2 \sin mx - \dfrac{2}{m}\int x \sin mx\,dx$ $(\star)$. Next let $U = x$, $dV = \sin mx\,dx \Rightarrow dU = dx$,

$V = -\dfrac{1}{m}\cos mx$, so $\int x \sin mx\,dx = -\dfrac{1}{m}x\cos mx + \dfrac{1}{m}\int \cos mx\,dx = -\dfrac{1}{m}x\cos mx + \dfrac{1}{m^2}\sin mx + C_1$.

Substituting for $\int x \sin mx\,dx$ in $(\star)$, we get

$$I = \frac{1}{m}x^2 \sin mx - \frac{2}{m}\left(-\frac{1}{m}x\cos mx + \frac{1}{m^2}\sin mx + C_1\right) = \frac{1}{m}x^2\sin mx + \frac{2}{m^2}x\cos mx - \frac{2}{m^3}\sin mx + C,$$

where $C = -\dfrac{2}{m}C_1$.

9. Let $u = \ln(2x+1)$, $dv = dx \Rightarrow du = \dfrac{2}{2x+1}\,dx$, $v = x$. Then

$$\begin{aligned}\int \ln(2x+1)\,dx &= x\ln(2x+1) - \int \frac{2x}{2x+1}\,dx = x\ln(2x+1) - \int \frac{(2x+1)-1}{2x+1}\,dx \\ &= x\ln(2x+1) - \int\left(1 - \frac{1}{2x+1}\right)dx = x\ln(2x+1) - x + \tfrac{1}{2}\ln(2x+1) + C \\ &= \tfrac{1}{2}(2x+1)\ln(2x+1) - x + C\end{aligned}$$

10. Let $u = \sin^{-1}x$, $dv = dx \Rightarrow du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then $\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \dfrac{x}{\sqrt{1-x^2}}\,dx$. Setting

$t = 1 - x^2$, we get $dt = -2x\,dx$, so $-\int \dfrac{x\,dx}{\sqrt{1-x^2}} = -\int t^{-1/2}\left(-\tfrac{1}{2}\,dt\right) = \tfrac{1}{2}(2t^{1/2}) + C = t^{1/2} + C = \sqrt{1-x^2} + C$.

Hence, $\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$.

11. Let $u = \arctan 4t$, $dv = dt \Rightarrow du = \dfrac{4}{1+(4t)^2}\,dt = \dfrac{4}{1+16t^2}\,dt$, $v = t$. Then

$\int \arctan 4t\,dt = t\arctan 4t - \int \dfrac{4t}{1+16t^2}\,dt = t\arctan 4t - \dfrac{1}{8}\int \dfrac{32t}{1+16t^2}\,dt = t\arctan 4t - \tfrac{1}{8}\ln(1+16t^2) + C$.

12. Let $u = \ln p$, $dv = p^5\,dp \Rightarrow du = \dfrac{1}{p}\,dp$, $v = \tfrac{1}{6}p^6$. Then $\int p^5 \ln p\,dp = \tfrac{1}{6}p^6 \ln p - \tfrac{1}{6}\int p^5\,dp = \tfrac{1}{6}p^6\ln p - \tfrac{1}{36}p^6 + C$.

13. Let $u = t$, $dv = \sec^2 2t\,dt \Rightarrow du = dt$, $v = \tfrac{1}{2}\tan 2t$. Then

$\int t\sec^2 2t\,dt = \tfrac{1}{2}t\tan 2t - \tfrac{1}{2}\int \tan 2t\,dt = \tfrac{1}{2}t\tan 2t - \tfrac{1}{4}\ln|\sec 2t| + C$.

14. Let $u = s$, $dv = 2^s\,ds \Rightarrow du = ds$, $v = \dfrac{1}{\ln 2}2^s$. Then

$\int s\,2^s\,ds = \dfrac{1}{\ln 2}s\,2^s - \dfrac{1}{\ln 2}\int 2^s\,ds = \dfrac{1}{\ln 2}s\,2^s - \dfrac{1}{(\ln 2)^2}2^s + C$ $\left[\text{or } \dfrac{2^s}{(\ln 2)^2}(s\ln 2 - 1) + C\right]$.

15. First let $u = (\ln x)^2$, $dv = dx \Rightarrow du = 2\ln x \cdot \frac{1}{x}\,dx$, $v = x$. Then by Equation 2,

$I = \int(\ln x)^2\,dx = x(\ln x)^2 - 2\int x\ln x \cdot \frac{1}{x}\,dx = x(\ln x)^2 - 2\int \ln x\,dx$. Next let $U = \ln x$, $dV = dx \Rightarrow$

$dU = 1/x\,dx$, $V = x$ to get $\int \ln x\,dx = x\ln x - \int x \cdot (1/x)\,dx = x\ln x - \int dx = x\ln x - x + C_1$. Thus,

$I = x(\ln x)^2 - 2(x\ln x - x + C_1) = x(\ln x)^2 - 2x\ln x + 2x + C$, where $C = -2C_1$.

16. Let $u = t$, $dv = \sinh mt\,dt \Rightarrow du = dt$, $v = \dfrac{1}{m}\cosh mt$. Then

$$\int t\sinh mt\,dt = \frac{1}{m}t\cosh mt - \int \frac{1}{m}\cosh mt\,dt = \frac{1}{m}t\cosh mt - \frac{1}{m^2}\sinh mt + C \quad [m \neq 0].$$

17. First let $u = \sin 3\theta$, $dv = e^{2\theta}\,d\theta \Rightarrow du = 3\cos 3\theta\,d\theta$, $v = \frac{1}{2}e^{2\theta}$. Then

$I = \int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U = \cos 3\theta$, $dV = e^{2\theta}\,d\theta \Rightarrow dU = -3\sin 3\theta\,d\theta$,

$V = \frac{1}{2}e^{2\theta}$ to get $\int e^{2\theta}\cos 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\cos 3\theta + \frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives

$I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}I \Rightarrow$

$\frac{13}{4}I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta + C_1$. Hence, $I = \frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$, where $C = \frac{4}{13}C_1$.

18. First let $u = e^{-\theta}$, $dv = \cos 2\theta\,d\theta \Rightarrow du = -e^{-\theta}\,d\theta$, $v = \frac{1}{2}\sin 2\theta$. Then

$I = \int e^{-\theta}\cos 2\theta\,d\theta = \frac{1}{2}e^{-\theta}\sin 2\theta - \int \frac{1}{2}\sin 2\theta\left(-e^{-\theta}\,d\theta\right) = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\int e^{-\theta}\sin 2\theta\,d\theta$.

Next let $U = e^{-\theta}$, $dV = \sin 2\theta\,d\theta \Rightarrow dU = -e^{-\theta}\,d\theta$, $V = -\frac{1}{2}\cos 2\theta$, so

$\int e^{-\theta}\sin 2\theta\,d\theta = -\frac{1}{2}e^{-\theta}\cos 2\theta - \int\left(-\frac{1}{2}\right)\cos 2\theta\left(-e^{-\theta}\,d\theta\right) = -\frac{1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\,d\theta$.

So $I = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\left[\left(-\frac{1}{2}e^{-\theta}\cos 2\theta\right) - \frac{1}{2}I\right] = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}I \Rightarrow$

$\frac{5}{4}I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1 \Rightarrow I = \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1\right) = \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$.

19. Let $u = t$, $dv = \sin 3t\,dt \Rightarrow du = dt$, $v = -\frac{1}{3}\cos 3t$. Then

$\int_0^\pi t\sin 3t\,dt = \left[-\frac{1}{3}t\cos 3t\right]_0^\pi + \frac{1}{3}\int_0^\pi \cos 3t\,dt = \left(\frac{1}{3}\pi - 0\right) + \frac{1}{9}\left[\sin 3t\right]_0^\pi = \frac{\pi}{3}$.

20. First let $u = x^2 + 1$, $dv = e^{-x}\,dx \Rightarrow du = 2x\,dx$, $v = -e^{-x}$. By (6),

$\int_0^1 (x^2+1)e^{-x}\,dx = \left[-(x^2+1)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\,dx$.

Next let $U = x$, $dV = e^{-x}\,dx \Rightarrow dU = dx$, $V = -e^{-x}$. By (6) again,

$\int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1$. So

$\int_0^1 (x^2+1)e^{-x}\,dx = -2e^{-1} + 1 + 2(-2e^{-1} + 1) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3$.

21. Let $u = t$, $dv = \cosh t\,dt \Rightarrow du = dt$, $v = \sinh t$. Then

$$\int_0^1 t\cosh t\,dt = \left[t\sinh t\right]_0^1 - \int_0^1 \sinh t\,dt = (\sinh 1 - \sinh 0) - \left[\cosh t\right]_0^1 = \sinh 1 - (\cosh 1 - \cosh 0)$$
$$= \sinh 1 - \cosh 1 + 1.$$

We can use the definitions of sinh and cosh to write the answer in terms of e:

$\sinh 1 - \cosh 1 + 1 = \frac{1}{2}(e^1 - e^{-1}) - \frac{1}{2}(e^1 + e^{-1}) + 1 = -e^{-1} + 1 = 1 - 1/e$.

22. Let $u = \ln y$, $dv = \dfrac{1}{\sqrt{y}}\,dy = y^{-1/2}\,dy \Rightarrow du = \dfrac{1}{y}\,dy$, $v = 2y^{1/2}$. Then

$$\int_4^9 \frac{\ln y}{\sqrt{y}}\,dy = \left[2\sqrt{y}\ln y\right]_4^9 - \int_4^9 2y^{-1/2}\,dy = (6\ln 9 - 4\ln 4) - \left[4\sqrt{y}\right]_4^9 = 6\ln 9 - 4\ln 4 - (12 - 8)$$
$$= 6\ln 9 - 4\ln 4 - 4$$

23. Let $u = \ln x$, $dv = x^{-2}\,dx \;\Rightarrow\; du = \dfrac{1}{x}\,dx$, $v = -x^{-1}$. By (6),

$$\int_1^2 \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 x^{-2}\,dx = -\tfrac{1}{2}\ln 2 + \ln 1 + \left[-\frac{1}{x}\right]_1^2 = -\tfrac{1}{2}\ln 2 + 0 - \tfrac{1}{2} + 1 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2.$$

24. First let $u = x^3$, $dv = \cos x\,dx \;\Rightarrow\; du = 3x^2\,dx$, $v = \sin x$. Then $I_1 = \int x^3 \cos x\,dx = x^3 \sin x - 3\int x^2 \sin x\,dx$. Next let $u_1 = x^2$, $dv_1 = \sin x\,dx \;\Rightarrow\; du_1 = 2x\,dx$, $v_1 = -\cos x$. Then $I_2 = \int x^2 \sin x\,dx = -x^2 \cos x + 2\int x \cos x\,dx$. Finally, let $u_2 = x$, $dv_2 = \cos x\,dx \;\Rightarrow\; du_2 = dx$, $v_2 = \sin x$. Then $\int x \cos x\,dx = x \sin x - \int \sin x\,dx = x \sin x + \cos x + C$. Substituting in the expression for I_2, we get $I_2 = -x^2 \cos x + 2(x \sin x + \cos x + C) = -x^2 \cos x + 2x \sin x + 2\cos x + 2C$. Substituting the last expression for I_2 into I_1 gives $I_1 = x^3 \sin x - 3(-x^2 \cos x + 2x \sin x + 2\cos x + 2C) = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x - 6C$. Thus,

$$\begin{aligned}\textstyle\int_0^\pi x^3 \cos x\,dx &= \left[x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x - 6C\right]_0^\pi \\ &= (0 - 3\pi^2 - 0 + 6 - 6C) - (0 + 0 - 0 - 6 - 6C) = 12 - 3\pi^2\end{aligned}$$

25. Let $u = y$, $dv = \dfrac{dy}{e^{2y}} = e^{-2y}dy \;\Rightarrow\; du = dy$, $v = -\tfrac{1}{2}e^{-2y}$. Then

$$\int_0^1 \frac{y}{e^{2y}}\,dy = \left[-\tfrac{1}{2}ye^{-2y}\right]_0^1 + \tfrac{1}{2}\int_0^1 e^{-2y}dy = \left(-\tfrac{1}{2}e^{-2} + 0\right) - \tfrac{1}{4}\left[e^{-2y}\right]_0^1 = -\tfrac{1}{2}e^{-2} - \tfrac{1}{4}e^{-2} + \tfrac{1}{4} = \tfrac{1}{4} - \tfrac{3}{4}e^{-2}.$$

26. Let $u = \arctan(1/x)$, $dv = dx \;\Rightarrow\; du = \dfrac{1}{1 + (1/x)^2} \cdot \dfrac{-1}{x^2}\,dx = \dfrac{-dx}{x^2 + 1}$, $v = x$. Then

$$\begin{aligned}\int_1^{\sqrt{3}} \arctan\!\left(\frac{1}{x}\right)dx &= \left[x \arctan\!\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x\,dx}{x^2 + 1} = \sqrt{3}\,\frac{\pi}{6} - 1 \cdot \frac{\pi}{4} + \frac{1}{2}\left[\ln(x^2 + 1)\right]_1^{\sqrt{3}} \\ &= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2) = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}\ln\frac{4}{2} = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}\ln 2\end{aligned}$$

27. Let $u = \cos^{-1} x$, $dv = dx \;\Rightarrow\; du = -\dfrac{dx}{\sqrt{1 - x^2}}$, $v = x$. Then

$$I = \int_0^{1/2} \cos^{-1} x\,dx = \left[x \cos^{-1} x\right]_0^{1/2} + \int_0^{1/2} \frac{x\,dx}{\sqrt{1 - x^2}} = \tfrac{1}{2} \cdot \tfrac{\pi}{3} + \int_1^{3/4} t^{-1/2}\left[-\tfrac{1}{2}dt\right],\ \text{where } t = 1 - x^2 \;\Rightarrow$$

$dt = -2x\,dx$. Thus, $I = \frac{\pi}{6} + \frac{1}{2}\int_{3/4}^1 t^{-1/2}\,dt = \frac{\pi}{6} + \left[\sqrt{t}\right]_{3/4}^1 = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2} = \frac{1}{6}\left(\pi + 6 - 3\sqrt{3}\right)$.

28. Let $u = (\ln x)^2$, $dv = x^{-3}\,dx \;\Rightarrow\; du = \dfrac{2\ln x}{x}\,dx$, $v = -\tfrac{1}{2}x^{-2}$. Then

$$I = \int_1^2 \frac{(\ln x)^2}{x^3}\,dx = \left[-\frac{(\ln x)^2}{2x^2}\right]_1^2 + \int_1^2 \frac{\ln x}{x^3}\,dx.\ \text{Now let } U = \ln x,\ dV = x^{-3}\,dx \;\Rightarrow\; dU = \frac{1}{x}\,dx,\ V = -\tfrac{1}{2}x^{-2}.$$

Then

$$\int_1^2 \frac{\ln x}{x^3}\,dx = \left[-\frac{\ln x}{2x^2}\right]_1^2 + \frac{1}{2}\int_1^2 x^{-3}dx = -\frac{1}{8}\ln 2 + 0 + \frac{1}{2}\left[-\frac{1}{2x^2}\right]_1^2 = -\frac{1}{8}\ln 2 + \frac{1}{2}\left(-\frac{1}{8} + \frac{1}{2}\right) = \frac{3}{16} - \frac{1}{8}\ln 2.$$

Thus $I = \left(-\frac{1}{8}(\ln 2)^2 + 0\right) + \left(\frac{3}{16} - \frac{1}{8}\ln 2\right) = -\frac{1}{8}(\ln 2)^2 - \frac{1}{8}\ln 2 + \frac{3}{16}$.

29. Let $u = \ln(\sin x)$, $dv = \cos x\,dx \;\Rightarrow\; du = \dfrac{\cos x}{\sin x}\,dx$, $v = \sin x$. Then

$I = \int \cos x \ln(\sin x)\,dx = \sin x \ln(\sin x) - \int \cos x\,dx = \sin x \ln(\sin x) - \sin x + C$.

Another method: Substitute $t = \sin x$, so $dt = \cos x\,dx$. Then $I = \int \ln t\,dt = t\ln t - t + C$ (see Example 2) and so

$I = \sin x\,(\ln \sin x - 1) + C$.

30. Let $u = r^2$, $dv = \dfrac{r}{\sqrt{4+r^2}}\,dr \;\Rightarrow\; du = 2r\,dr$, $v = \sqrt{4+r^2}$. By (6),

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \left[r^2\sqrt{4+r^2}\right]_0^1 - 2\int_0^1 r\sqrt{4+r^2}\,dr = \sqrt{5} - \tfrac{2}{3}\left[(4+r^2)^{3/2}\right]_0^1$$
$$= \sqrt{5} - \tfrac{2}{3}(5)^{3/2} + \tfrac{2}{3}(8) = \sqrt{5}\left(1 - \tfrac{10}{3}\right) + \tfrac{16}{3} = \tfrac{16}{3} - \tfrac{7}{3}\sqrt{5}$$

31. Let $u = (\ln x)^2$, $dv = x^4\,dx \;\Rightarrow\; du = 2\dfrac{\ln x}{x}\,dx$, $v = \dfrac{x^5}{5}$. By (6),

$$\int_1^2 x^4(\ln x)^2\,dx = \left[\frac{x^5}{5}(\ln x)^2\right]_1^2 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx = \tfrac{32}{5}(\ln 2)^2 - 0 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx.$$

Let $U = \ln x$, $dV = \dfrac{x^4}{5}\,dx \;\Rightarrow\; dU = \dfrac{1}{x}\,dx$, $V = \dfrac{x^5}{25}$.

Then $\displaystyle\int_1^2 \frac{x^4}{5}\ln x\,dx = \left[\frac{x^5}{25}\ln x\right]_1^2 - \int_1^2 \frac{x^4}{25}\,dx = \tfrac{32}{25}\ln 2 - 0 - \left[\frac{x^5}{125}\right]_1^2 = \tfrac{32}{25}\ln 2 - \left(\tfrac{32}{125} - \tfrac{1}{125}\right).$

So $\int_1^2 x^4(\ln x)^2\,dx = \tfrac{32}{5}(\ln 2)^2 - 2\left(\tfrac{32}{25}\ln 2 - \tfrac{31}{125}\right) = \tfrac{32}{5}(\ln 2)^2 - \tfrac{64}{25}\ln 2 + \tfrac{62}{125}$.

32. Let $u = \sin(t-s)$, $dv = e^s\,ds \;\Rightarrow\; du = -\cos(t-s)\,ds$, $v = e^s$. Then

$I = \int_0^t e^s \sin(t-s)\,ds = \left[e^s\sin(t-s)\right]_0^t + \int_0^t e^s\cos(t-s)\,ds = e^t\sin 0 - e^0\sin t + I_1$. For I_1, let $U = \cos(t-s)$,

$dV = e^s\,ds \;\Rightarrow\; dU = \sin(t-s)\,ds$, $V = e^s$. So $I_1 = \left[e^s\cos(t-s)\right]_0^t - \int_0^t e^s\sin(t-s)\,ds = e^t\cos 0 - e^0\cos t - I$.

Thus, $I = -\sin t + e^t - \cos t - I \;\Rightarrow\; 2I = e^t - \cos t - \sin t \;\Rightarrow\; I = \tfrac{1}{2}(e^t - \cos t - \sin t)$.

33. Let $y = \sqrt{x}$, so that $dy = \tfrac{1}{2}x^{-1/2}\,dx = \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{2y}\,dx$. Thus, $\int \cos\sqrt{x}\,dx = \int \cos y\,(2y\,dy) = 2\int y\cos y\,dy$. Now

use parts with $u = y$, $dv = \cos y\,dy$, $du = dy$, $v = \sin y$ to get $\int y\cos y\,dy = y\sin y - \int \sin y\,dy = y\sin y + \cos y + C_1$,

so $\int \cos\sqrt{x}\,dx = 2y\sin y + 2\cos y + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$.

34. Let $x = -t^2$, so that $dx = -2t\,dt$. Thus, $\int t^3 e^{-t^2}\,dt = \int (-t^2)e^{-t^2}\left(\tfrac{1}{2}\right)(-2t\,dt) = \tfrac{1}{2}\int xe^x\,dx$. Now use parts with

$u = x, dv = e^x\,dx, du = dx, v = e^x$ to get

$\tfrac{1}{2}\int xe^x\,dx = \tfrac{1}{2}\left(xe^x - \int e^x\,dx\right) = \tfrac{1}{2}xe^x - \tfrac{1}{2}e^x + C = -\tfrac{1}{2}(1-x)e^x + C = -\tfrac{1}{2}(1+t^2)e^{-t^2} + C$.

35. Let $x = \theta^2$, so that $dx = 2\theta\,d\theta$. Thus, $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)\,d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^2\cos(\theta^2)\cdot\tfrac{1}{2}(2\theta\,d\theta) = \tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx$. Now use

parts with $u = x$, $dv = \cos x\,dx$, $du = dx$, $v = \sin x$ to get

$$\tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx = \tfrac{1}{2}\left(\left[x\sin x\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\sin x\,dx\right) = \tfrac{1}{2}\left[x\sin x + \cos x\right]_{\pi/2}^{\pi}$$
$$= \tfrac{1}{2}(\pi\sin\pi + \cos\pi) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\right) = \tfrac{1}{2}(\pi\cdot 0 - 1) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\cdot 1 + 0\right) = -\tfrac{1}{2} - \tfrac{\pi}{4}$$

36. Let $x = \cos t$, so that $dx = -\sin t\,dt$. Thus,

$\int_0^\pi e^{\cos t} \sin 2t\,dt = \int_0^\pi e^{\cos t}(2\sin t\,\cos t)\,dt = \int_1^{-1} e^x \cdot 2x\,(-dx) = 2\int_{-1}^1 xe^x\,dx$. Now use parts with $u = x$, $dv = e^x\,dx$, $du = dx$, $v = e^x$ to get

$2\int_{-1}^1 xe^x\,dx = 2\left(\left[xe^x\right]_{-1}^1 - \int_{-1}^1 e^x\,dx\right) = 2\left(e^1 + e^{-1} - \left[e^x\right]_{-1}^1\right) = 2(e + e^{-1} - [e^1 - e^{-1}]) = 2(2e^{-1}) = 4/e.$

37. Let $y = 1 + x$, so that $dy = dx$. Thus, $\int x\ln(1+x)\,dx = \int (y-1)\ln y\,dy$. Now use parts with $u = \ln y$, $dv = (y-1)\,dy$, $du = \frac{1}{y}\,dy$, $v = \frac{1}{2}y^2 - y$ to get

$$\begin{aligned}\int (y-1)\ln y\,dy &= \left(\tfrac{1}{2}y^2 - y\right)\ln y - \int\left(\tfrac{1}{2}y - 1\right)dy = \tfrac{1}{2}y(y-2)\ln y - \tfrac{1}{4}y^2 + y + C \\ &= \tfrac{1}{2}(1+x)(x-1)\ln(1+x) - \tfrac{1}{4}(1+x)^2 + 1 + x + C,\end{aligned}$$

which can be written as $\frac{1}{2}(x^2-1)\ln(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{4} + C$.

38. Let $y = \ln x$, so that $dy = \frac{1}{x}\,dx \;\Rightarrow\; dx = x\,dy = e^y\,dy$. Thus,

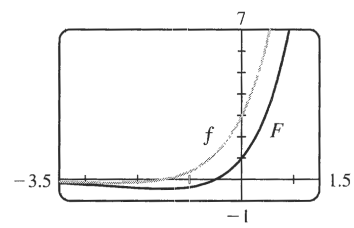
$\int \sin(\ln x)\,dx = \int \sin y\,e^y\,dy = \frac{1}{2}e^y(\sin y - \cos y) + C$ [by Example 4] $= \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C$.

In Exercises 39 – 42, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

39. Let $u = 2x + 3$, $dv = e^x\,dx \;\Rightarrow\; du = 2\,dx$, $v = e^x$. Then

$$\begin{aligned}\int (2x+3)e^x\,dx &= (2x+3)e^x - 2\int e^x\,dx = (2x+3)e^x - 2e^x + C \\ &= (2x+1)\,e^x + C\end{aligned}$$

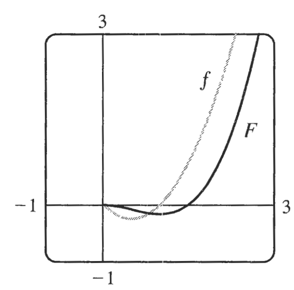
We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



40. Let $u = \ln x$, $dv = x^{3/2}\,dx \;\Rightarrow\; du = \frac{1}{x}\,dx$, $v = \frac{2}{5}x^{5/2}$. Then

$$\begin{aligned}\int x^{3/2}\ln x\,dx &= \tfrac{2}{5}x^{5/2}\ln x - \tfrac{2}{5}\int x^{3/2}\,dx = \tfrac{2}{5}x^{5/2}\ln x - \left(\tfrac{2}{5}\right)^2 x^{5/2} + C \\ &= \tfrac{2}{5}x^{5/2}\ln x - \tfrac{4}{25}x^{5/2} + C\end{aligned}$$

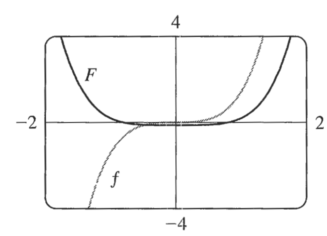
We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



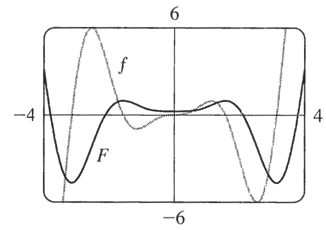
41. Let $u = \frac{1}{2}x^2$, $dv = 2x\sqrt{1+x^2}\,dx \;\Rightarrow\; du = x\,dx$, $v = \frac{2}{3}(1+x^2)^{3/2}$.

Then

$$\begin{aligned}\int x^3\sqrt{1+x^2}\,dx &= \tfrac{1}{2}x^2\left[\tfrac{2}{3}(1+x^2)^{3/2}\right] - \tfrac{2}{3}\int x(1+x^2)^{3/2}dx \\ &= \tfrac{1}{3}x^2(1+x^2)^{3/2} - \tfrac{2}{3}\cdot\tfrac{2}{5}\cdot\tfrac{1}{2}(1+x^2)^{5/2} + C \\ &= \tfrac{1}{3}x^2(1+x^2)^{3/2} - \tfrac{2}{15}(1+x^2)^{5/2} + C\end{aligned}$$



Another method: Use substitution with $u = 1 + x^2$ to get $\frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C$.



42. First let $u = x^2$, $dv = \sin 2x\,dx \Rightarrow du = 2x\,dx$, $v = -\frac{1}{2}\cos 2x$.

Then $I = \int x^2 \sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx$.

Next let $U = x$, $dV = \cos 2x\,dx \Rightarrow dU = dx$, $V = \frac{1}{2}\sin 2x$, so

$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$.

Thus, $I = -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$.

We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

43. (a) Take $n = 2$ in Example 6 to get $\displaystyle\int \sin^2 x\,dx = -\frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

(b) $\int \sin^4 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{4}\int \sin^2 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$.

44. (a) Let $u = \cos^{n-1} x$, $dv = \cos x\,dx \Rightarrow du = -(n-1)\cos^{n-2} x\,\sin x\,dx$, $v = \sin x$ in (2):

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,\sin^2 x\,dx\\ &= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,(1-\cos^2 x)dx\\ &= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx\end{aligned}$$

Rearranging terms gives $n\int \cos^n x\,dx = \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx$ or

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\,\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

(b) Take $n = 2$ in part (a) to get $\int \cos^2 x\,dx = \frac{1}{2}\cos x\,\sin x + \frac{1}{2}\int 1\,dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4} + C$.

(c) $\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x\,\sin x + \frac{3}{4}\int \cos^2 x\,dx = \frac{1}{4}\cos^3 x\,\sin x + \frac{3}{8}x + \frac{3}{16}\sin 2x + C$

45. (a) From Example 6, $\displaystyle\int \sin^n x\,dx = -\frac{1}{n}\cos x\,\sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$. Using (6),

$$\begin{aligned}\int_0^{\pi/2} \sin^n x\,dx &= \left[-\frac{\cos x\,\sin^{n-1} x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx\\ &= (0-0) + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx\end{aligned}$$

(b) Using $n = 3$ in part (a), we have $\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \left[-\frac{2}{3}\cos x\right]_0^{\pi/2} = \frac{2}{3}$.

Using $n = 5$ in part (a), we have $\int_0^{\pi/2} \sin^5 x\,dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$.

(c) The formula holds for $n = 1$ (that is, $2n + 1 = 3$) by (b). Assume it holds for some $k \geq 1$. Then

$\displaystyle\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$. By Example 6,

$$\begin{aligned}\int_0^{\pi/2} \sin^{2k+3} x\,dx &= \frac{2k+2}{2k+3}\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2k+2}{2k+3}\cdot\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}\\ &= \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)[2(k+1)]}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)[2(k+1)+1]},\end{aligned}$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

46. Using Exercise 45(a), we see that the formula holds for $n = 1$, because $\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} 1\,dx = \frac{1}{2}\left[x\right]_0^{\pi/2} = \frac{1}{2} \cdot \frac{\pi}{2}$.

Now assume it holds for some $k \geq 1$. Then $\displaystyle\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2k)} \frac{\pi}{2}$. By Exercise 45(a),

$$\begin{aligned}\int_0^{\pi/2} \sin^{2(k+1)} x\,dx &= \frac{2k+1}{2k+2}\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{2k+1}{2k+2} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2k)} \frac{\pi}{2} \\ &= \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)(2k+1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2k)(2k+2)} \cdot \frac{\pi}{2},\end{aligned}$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

47. Let $u = (\ln x)^n$, $dv = dx \Rightarrow du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2,

$\int (\ln x)^n\,dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$.

48. Let $u = x^n$, $dv = e^x\,dx \Rightarrow du = nx^{n-1}\,dx$, $v = e^x$. By Equation 2, $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$.

49. $\int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx = \int \tan^{n-2} x\,(\sec^2 x - 1)\,dx = \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$

$= I - \int \tan^{n-2} x\,dx$.

Let $u = \tan^{n-2} x$, $dv = \sec^2 x\,dx \Rightarrow du = (n-2)\tan^{n-3} x \sec^2 x\,dx$, $v = \tan x$. Then, by Equation 2,

$$\begin{aligned} I &= \tan^{n-1} x - (n-2)\int \tan^{n-2} x \sec^2 x\,dx \\ 1I &= \tan^{n-1} x - (n-2)I \\ (n-1)I &= \tan^{n-1} x \\ I &= \frac{\tan^{n-1} x}{n-1}\end{aligned}$$

Returning to the original integral, $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$.

50. Let $u = \sec^{n-2} x$, $dv = \sec^2 x\,dx \Rightarrow du = (n-2)\sec^{n-3} x \sec x \tan x\,dx$, $v = \tan x$. Then, by Equation 2,

$$\begin{aligned}\int \sec^n x\,dx &= \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx \\ &= \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x\,(\sec^2 x - 1)\,dx \\ &= \tan x \sec^{n-2} x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx\end{aligned}$$

so $(n-1)\int \sec^n x\,dx = \tan x \sec^{n-2} x + (n-2)\int \sec^{n-2} x\,dx$. If $n - 1 \neq 0$, then

$\displaystyle\int \sec^n x\,dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$.

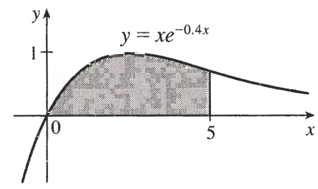
51. By repeated applications of the reduction formula in Exercise 47,

$$\begin{aligned}\int (\ln x)^3\,dx &= x(\ln x)^3 - 3\int (\ln x)^2\,dx = x(\ln x)^3 - 3\left[x(\ln x)^2 - 2\int (\ln x)^1\,dx\right] \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6\left[x(\ln x)^1 - 1\int (\ln x)^0\,dx\right] \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6\int 1\,dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C\end{aligned}$$

52. By repeated applications of the reduction formula in Exercise 48,

$$\begin{aligned}\int x^4e^x\,dx &= x^4e^x - 4\int x^3e^x\,dx = x^4e^x - 4\left(x^3e^x - 3\int x^2e^x\,dx\right)\\ &= x^4e^x - 4x^3e^x + 12\left(x^2e^x - 2\int x^1e^x\,dx\right) = x^4e^x - 4x^3e^x + 12x^2e^x - 24\left(x^1e^x - \int x^0e^x\,dx\right)\\ &= x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C \quad \left[\text{or } e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C\right]\end{aligned}$$

53. Area $= \int_0^5 xe^{-0.4x}dx$. Let $u = x$, $dv = e^{-0.4x}\,dx \Rightarrow$ $du = dx$, $v = -2.5e^{-0.4x}$. Then

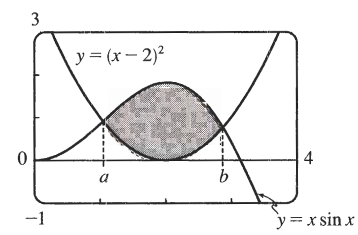


$$\begin{aligned}\text{area} &= \left[-2.5xe^{-0.4x}\right]_0^5 + 2.5\int_0^5 e^{-0.4x}\,dx\\ &= -12.5e^{-2} + 0 + 2.5\left[-2.5e^{-0.4x}\right]_0^5\\ &= -12.5e^{-2} - 6.25(e^{-2} - 1) = 6.25 - 18.75e^{-2} \text{ or } \tfrac{25}{4} - \tfrac{75}{4}e^{-2}\end{aligned}$$

54. The curves $y = x\ln x$ and $y = 5\ln x$ intersect when $x\ln x = 5\ln x \Leftrightarrow x\ln x - 5\ln x = 0 \Leftrightarrow (x-5)\ln x = 0$; that is, when $x = 1$ or $x = 5$. For $1 < x < 5$, we have $5\ln x > x\ln x$ since $\ln x > 0$. Thus, area $= \int_1^5 (5\ln x - x\ln x)\,dx = \int_1^5 [(5-x)\ln x]\,dx$. Let $u = \ln x$, $dv = (5-x)\,dx \Rightarrow du = dx/x$, $v = 5x - \frac{1}{2}x^2$. Then

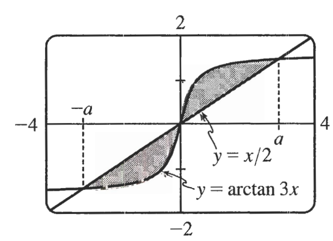
$$\begin{aligned}\text{area} &= \left[(\ln x)\left(5x - \tfrac{1}{2}x^2\right)\right]_1^5 - \int_1^5 \left[\left(5x - \tfrac{1}{2}x^2\right)\tfrac{1}{x}\right]dx = (\ln 5)\left(\tfrac{25}{2}\right) - 0 - \int_1^5 \left(5 - \tfrac{1}{2}x\right)dx\\ &= \tfrac{25}{2}\ln 5 - \left[5x - \tfrac{1}{4}x^2\right]_1^5 = \tfrac{25}{2}\ln 5 - \left[\left(25 - \tfrac{25}{4}\right) - \left(5 - \tfrac{1}{4}\right)\right] = \tfrac{25}{2}\ln 5 - 14\end{aligned}$$

55. The curves $y = x\sin x$ and $y = (x-2)^2$ intersect at $a \approx 1.04748$ and $b \approx 2.87307$, so



$$\begin{aligned}\text{area} &= \int_a^b \left[x\sin x - (x-2)^2\right]dx\\ &= \left[-x\cos x + \sin x - \tfrac{1}{3}(x-2)^3\right]_a^b \quad \text{[by Example 1]}\\ &\approx 2.81358 - 0.63075 = 2.18283\end{aligned}$$

56. The curves $y = \arctan 3x$ and $y = \frac{1}{2}x$ intersect at $x = \pm a \approx \pm 2.91379$, so



$$\begin{aligned}\text{area} &= \int_{-a}^a \left|\arctan 3x - \tfrac{1}{2}x\right|dx = 2\int_0^a \left(\arctan 3x - \tfrac{1}{2}x\right)dx\\ &= 2\left[x\arctan 3x - \tfrac{1}{6}\ln(1+9x^2) - \tfrac{1}{4}x^2\right]_0^a \quad \text{[see Example 5]}\\ &\approx 2(1.39768) = 2.79536\end{aligned}$$

57. $V = \int_0^1 2\pi x\cos(\pi x/2)\,dx$. Let $u = x$, $dv = \cos(\pi x/2)\,dx \Rightarrow du = dx$, $v = \frac{2}{\pi}\sin(\pi x/2)$.

$$V = 2\pi\left[\frac{2}{\pi}x\sin\left(\frac{\pi x}{2}\right)\right]_0^1 - 2\pi\cdot\frac{2}{\pi}\int_0^1 \sin\left(\frac{\pi x}{2}\right)dx = 2\pi\left(\frac{2}{\pi} - 0\right) - 4\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_0^1 = 4 + \frac{8}{\pi}(0-1) = 4 - \frac{8}{\pi}.$$

58. Volume $= \int_0^1 2\pi x(e^x - e^{-x})\,dx = 2\pi\int_0^1 (xe^x - xe^{-x})\,dx = 2\pi\left[\int_0^1 xe^x\,dx - \int_0^1 xe^{-x}\,dx\right]$ [both integrals by parts]

$$= 2\pi\left[(xe^x - e^x) - \left(-xe^{-x} - e^{-x}\right)\right]_0^1 = 2\pi[2/e - 0] = 4\pi/e$$

59. Volume $= \int_{-1}^{0} 2\pi(1-x)e^{-x}\,dx$. Let $u = 1-x$, $dv = e^{-x}\,dx \;\Rightarrow\; du = -\,dx$, $v = -e^{-x}$.

$V = 2\pi\left[(1-x)(-e^{-x})\right]_{-1}^{0} - 2\pi\int_{-1}^{0} e^{-x}\,dx = 2\pi\left[(x-1)(e^{-x}) + e^{-x}\right]_{-1}^{0} = 2\pi\left[xe^{-x}\right]_{-1}^{0} = 2\pi(0+e) = 2\pi e$

60. Volume $= \int_1^{\pi} 2\pi y \cdot \ln y\,dy = 2\pi\left[\frac{1}{2}y^2\ln y - \frac{1}{4}y^2\right]_1^{\pi}$ [by parts with $u = \ln y$ and $dv = y\,dy$]

$$= 2\pi\left[\tfrac{1}{4}y^2(2\ln y - 1)\right]_1^{\pi} = 2\pi\left[\frac{\pi^2(2\ln\pi - 1)}{4} - \frac{(0-1)}{4}\right] = \pi^3\ln\pi - \frac{\pi^3}{2} + \frac{\pi}{2}$$

61. The average value of $f(x) = x^2\ln x$ on the interval $[1,3]$ is $f_{\text{ave}} = \dfrac{1}{3-1}\displaystyle\int_1^3 x^2\ln x\,dx = \frac{1}{2}I$.

Let $u = \ln x$, $dv = x^2\,dx \;\Rightarrow\; du = (1/x)\,dx$, $v = \frac{1}{3}x^3$.

So $I = \left[\frac{1}{3}x^3\ln x\right]_1^3 - \int_1^3 \frac{1}{3}x^2\,dx = (9\ln 3 - 0) - \left[\frac{1}{9}x^3\right]_1^3 = 9\ln 3 - \left(3 - \frac{1}{9}\right) = 9\ln 3 - \frac{26}{9}$.

Thus, $f_{\text{ave}} = \frac{1}{2}I = \frac{1}{2}\left(9\ln 3 - \frac{26}{9}\right) = \frac{9}{2}\ln 3 - \frac{13}{9}$.

62. The rocket will have height $H = \int_0^{60} v(t)\,dt$ after 60 seconds.

$$H = \int_0^{60}\left[-gt - v_e\ln\left(\frac{m-rt}{m}\right)\right]dt = -g\left[\tfrac{1}{2}t^2\right]_0^{60} - v_e\left[\int_0^{60}\ln(m-rt)\,dt - \int_0^{60}\ln m\,dt\right]$$

$$= -g(1800) + v_e(\ln m)(60) - v_e\int_0^{60}\ln(m-rt)\,dt$$

Let $u = \ln(m-rt)$, $dv = dt \;\Rightarrow\; du = \dfrac{1}{m-rt}(-r)\,dt$, $v = t$. Then

$$\int_0^{60}\ln(m-rt)\,dt = \Big[t\ln(m-rt)\Big]_0^{60} + \int_0^{60}\frac{rt}{m-rt}\,dt = 60\ln(m-60r) + \int_0^{60}\left(-1 + \frac{m}{m-rt}\right)dt$$

$$= 60\ln(m-60r) + \left[-t - \frac{m}{r}\ln(m-rt)\right]_0^{60} = 60\ln(m-60r) - 60 - \frac{m}{r}\ln(m-60r) + \frac{m}{r}\ln m$$

So $H = -1800g + 60v_e\ln m - 60v_e\ln(m-60r) + 60v_e + \dfrac{m}{r}v_e\ln(m-60r) - \dfrac{m}{r}v_e\ln m$. Substituting $g = 9.8$,

$m = 30{,}000$, $r = 160$, and $v_e = 3000$ gives us $H \approx 14{,}844$ m.

63. Since $v(t) > 0$ for all t, the desired distance is $s(t) = \int_0^t v(w)\,dw = \int_0^t w^2e^{-w}\,dw$.

First let $u = w^2$, $dv = e^{-w}\,dw \;\Rightarrow\; du = 2w\,dw$, $v = -e^{-w}$. Then $s(t) = \left[-w^2e^{-w}\right]_0^t + 2\int_0^t we^{-w}\,dw$.

Next let $U = w$, $dV = e^{-w}\,dw \;\Rightarrow\; dU = dw$, $V = -e^{-w}$. Then

$$s(t) = -t^2e^{-t} + 2\left(\left[-we^{-w}\right]_0^t + \int_0^t e^{-w}\,dw\right) = -t^2e^{-t} + 2\left(-te^{-t} + 0 + \left[-e^{-w}\right]_0^t\right)$$

$$= -t^2e^{-t} + 2(-te^{-t} - e^{-t} + 1) = -t^2e^{-t} - 2te^{-t} - 2e^{-t} + 2 = 2 - e^{-t}(t^2 + 2t + 2) \text{ meters}$$

64. Suppose $f(0) = g(0) = 0$ and let $u = f(x)$, $dv = g''(x)\,dx \;\Rightarrow\; du = f'(x)\,dx$, $v = g'(x)$.

Then $\int_0^a f(x)\,g''(x)\,dx = \Big[f(x)\,g'(x)\Big]_0^a - \int_0^a f'(x)\,g'(x)\,dx = f(a)\,g'(a) - \int_0^a f'(x)\,g'(x)\,dx$.

Now let $U = f'(x)$, $dV = g'(x)\,dx \;\Rightarrow\; dU = f''(x)\,dx$ and $V = g(x)$, so

$\int_0^a f'(x)\,g'(x)\,dx = \Big[f'(x)\,g(x)\Big]_0^a - \int_0^a f''(x)\,g(x)\,dx = f'(a)\,g(a) - \int_0^a f''(x)\,g(x)\,dx$.

Combining the two results, we get $\int_0^a f(x)\,g''(x)\,dx = f(a)\,g'(a) - f'(a)\,g(a) + \int_0^a f''(x)\,g(x)\,dx$.

65. For $I = \int_1^4 x f''(x)\,dx$, let $u = x$, $dv = f''(x)\,dx \quad \Rightarrow \quad du = dx$, $v = f'(x)$. Then

$I = \left[xf'(x)\right]_1^4 - \int_1^4 f'(x)\,dx = 4f'(4) - 1 \cdot f'(1) - [f(4) - f(1)] = 4 \cdot 3 - 1 \cdot 5 - (7 - 2) = 12 - 5 - 5 = 2.$

We used the fact that f'' is continuous to guarantee that I exists.

66. (a) Take $g(x) = x$ and $g'(x) = 1$ in Equation 1.

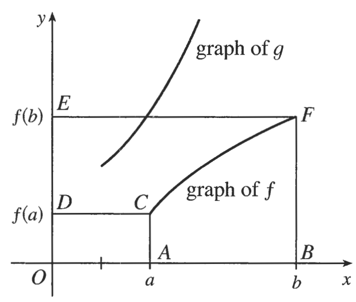
(b) By part (a), $\int_a^b f(x)\,dx = bf(b) - a\,f(a) - \int_a^b x\,f'(x)\,dx$. Now let $y = f(x)$, so that $x = g(y)$ and $dy = f'(x)\,dx$.

Then $\int_a^b x\,f'(x)\,dx = \int_{f(a)}^{f(b)} g(y)\,dy$. The result follows.

(c) Part (b) says that the area of region $ABFC$ is

$$= \quad bf(b) \quad - \quad af(a) \quad - \quad \int_{f(a)}^{f(b)} g(y)\,dy$$

= (area of rectangle $OBFE$) − (area of rectangle $OACD$) − (area of region $DCFE$)



(d) We have $f(x) = \ln x$, so $f^{-1}(x) = e^x$, and since $g = f^{-1}$, we have $g(y) = e^y$. By part (b),

$$\int_1^e \ln x\,dx = e\ln e - 1\ln 1 - \int_{\ln 1}^{\ln e} e^y\,dy = e - \int_0^1 e^y\,dy = e - \left[e^y\right]_0^1 = e - (e - 1) = 1.$$

67. Using the formula for volumes of rotation and the figure, we see that

Volume $= \int_0^d \pi b^2\,dy - \int_0^c \pi a^2\,dy - \int_c^d \pi[g(y)]^2\,dy = \pi b^2 d - \pi a^2 c - \int_c^d \pi[g(y)]^2\,dy$. Let $y = f(x)$,

which gives $dy = f'(x)\,dx$ and $g(y) = x$, so that $V = \pi b^2 d - \pi a^2 c - \pi \int_a^b x^2 f'(x)\,dx$.

Now integrate by parts with $u = x^2$, and $dv = f'(x)\,dx \quad \Rightarrow \quad du = 2x\,dx$, $v = f(x)$, and

$\int_a^b x^2\,f'(x)\,dx = \left[x^2\,f(x)\right]_a^b - \int_a^b 2x\,f(x)\,dx = b^2\,f(b) - a^2\,f(a) - \int_a^b 2x\,f(x)\,dx$, but $f(a) = c$ and $f(b) = d \quad \Rightarrow$

$V = \pi b^2 d - \pi a^2 c - \pi\left[b^2 d - a^2 c - \int_a^b 2x\,f(x)\,dx\right] = \int_a^b 2\pi x\,f(x)\,dx.$

68. (a) We note that for $0 \le x \le \frac{\pi}{2}$, $0 \le \sin x \le 1$, so $\sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x$. So by the second Comparison Property of the Integral, $I_{2n+2} \le I_{2n+1} \le I_{2n}$.

(b) Substituting directly into the result from Exercise 46, we get

$$\frac{I_{2n+2}}{I_{2n}} = \frac{\dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot [2(n+1) - 1]}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot [2(n+1)]} \dfrac{\pi}{2}}{\dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} \dfrac{\pi}{2}} = \frac{2(n+1) - 1}{2(n+1)} = \frac{2n+1}{2n+2}$$

(c) We divide the result from part (a) by I_{2n}. The inequalities are preserved since I_{2n} is positive: $\dfrac{I_{2n+2}}{I_{2n}} \le \dfrac{I_{2n+1}}{I_{2n}} \le \dfrac{I_{2n}}{I_{2n}}$.

Now from part (b), the left term is equal to $\dfrac{2n+1}{2n+2}$, so the expression becomes $\dfrac{2n+1}{2n+2} \le \dfrac{I_{2n+1}}{I_{2n}} \le 1$. Now

$\displaystyle\lim_{n\to\infty} \frac{2n+1}{2n+2} = \lim_{n\to\infty} 1 = 1$, so by the Squeeze Theorem, $\displaystyle\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$.

(d) We substitute the results from Exercises 45 and 46 into the result from part (c):

$$1 = \lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty} \frac{\dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}}{\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}\dfrac{\pi}{2}} = \lim_{n\to\infty} \left[\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}\right]\left[\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}\left(\frac{2}{\pi}\right)\right]$$

$$= \lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{2}{\pi} \qquad \text{[rearrange terms]}$$

Multiplying both sides by $\frac{\pi}{2}$ gives us the *Wallis product*:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots$$

(e) The area of the kth rectangle is k. At the $2n$th step, the area is increased from $2n-1$ to $2n$ by multiplying the width by $\dfrac{2n}{2n-1}$, and at the $(2n+1)$th step, the area is increased from $2n$ to $2n+1$ by multiplying the height by $\dfrac{2n+1}{2n}$. These two steps multiply the ratio of width to height by $\dfrac{2n}{2n-1}$ and $\dfrac{1}{(2n+1)/(2n)} = \dfrac{2n}{2n+1}$ respectively. So, by part (d), the limiting ratio is $\dfrac{2}{1}\cdot\dfrac{2}{3}\cdot\dfrac{4}{3}\cdot\dfrac{4}{5}\cdot\dfrac{6}{5}\cdot\dfrac{6}{7}\cdot\cdots = \dfrac{\pi}{2}$.