## 7.2 Trigonometric Integrals

The symbols $\overset{s}{=}$ and $\overset{c}{=}$ indicate the use of the substitutions $\{u = \sin x, du = \cos x\,dx\}$ and $\{u = \cos x, du = -\sin x\,dx\}$, respectively.

**1.** $\int \sin^3 x \cos^2 x\,dx = \int \sin^2 x \cos^2 x \sin x\,dx = \int (1 - \cos^2 x)\cos^2 x \sin x\,dx \overset{c}{=} \int (1-u^2)u^2(-du)$

$= \int (u^2 - 1)u^2\,du = \int (u^4 - u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

**2.** $\int \sin^6 x \cos^3 x\,dx = \int \sin^6 x \cos^2 x \cos x\,dx = \int \sin^6 x\,(1 - \sin^2 x)\cos x\,dx \overset{s}{=} \int u^6(1-u^2)\,du$

$= \int (u^6 - u^8)\,du = \frac{1}{7}u^7 - \frac{1}{9}u^9 + C = \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C$

**3.** $\int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x \cos^2 x \cos x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x\,(1 - \sin^2 x)\cos x\,dx \overset{s}{=} \int_1^{\sqrt{2}/2} u^5(1-u^2)\,du$

$= \int_1^{\sqrt{2}/2} (u^5 - u^7)\,du = \left[\frac{1}{6}u^6 - \frac{1}{8}u^8\right]_1^{\sqrt{2}/2} = \left(\frac{1/8}{6} - \frac{1/16}{8}\right) - \left(\frac{1}{6} - \frac{1}{8}\right) = -\frac{11}{384}$

**4.** $\int_0^{\pi/2} \cos^5 x\,dx = \int_0^{\pi/2} (\cos^2 x)^2 \cos x\,dx = \int_0^{\pi/2} (1 - \sin^2 x)^2 \cos x\,dx \overset{s}{=} \int_0^1 (1-u^2)^2\,du$

$= \int_0^1 (1 - 2u^2 + u^4)\,du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = \left(1 - \frac{2}{3} + \frac{1}{5}\right) - 0 = \frac{8}{15}$

**5.** Let $y = \pi x$, so $dy = \pi\, dx$ and

$$\begin{aligned}\int \sin^2(\pi x)\cos^5(\pi x)\,dx &= \tfrac{1}{\pi}\int \sin^2 y\,\cos^5 y\,dy = \tfrac{1}{\pi}\int \sin^2 y\,\cos^4 y\,\cos y\,dy \\ &= \tfrac{1}{\pi}\int \sin^2 y\,(1-\sin^2 y)^2\,\cos y\,dy \overset{s}{=} \tfrac{1}{\pi}\int u^2(1-u^2)^2\,du = \tfrac{1}{\pi}\int (u^2 - 2u^4 + u^6)\,du \\ &= \tfrac{1}{\pi}\left(\tfrac{1}{3}u^3 - \tfrac{2}{5}u^5 + \tfrac{1}{7}u^7\right) + C = \tfrac{1}{3\pi}\sin^3 y - \tfrac{2}{5\pi}\sin^5 y + \tfrac{1}{7\pi}\sin^7 y + C \\ &= \tfrac{1}{3\pi}\sin^3(\pi x) - \tfrac{2}{5\pi}\sin^5(\pi x) + \tfrac{1}{7\pi}\sin^7(\pi x) + C\end{aligned}$$

**6.** Let $y = \sqrt{x}$, so that $dy = \dfrac{1}{2\sqrt{x}}\,dx$ and $dx = 2y\,dy$. Then

$$\begin{aligned}\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}\,dx &= \int \frac{\sin^3 y}{y}(2y\,dy) = 2\int \sin^3 y\,dy = 2\int \sin^2 y\,\sin y\,dy = 2\int (1-\cos^2 y)\,\sin y\,dy \\ &\overset{c}{=} 2\int (1-u^2)(-du) = 2\int (u^2-1)\,du = 2\left(\tfrac{1}{3}u^3 - u\right) + C = \tfrac{2}{3}\cos^3 y - 2\cos y + C \\ &= \tfrac{2}{3}\cos^3(\sqrt{x}) - 2\cos\sqrt{x} + C\end{aligned}$$

**7.** $\int_0^{\pi/2}\cos^2\theta\,d\theta = \int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\,d\theta$ [half-angle identity]

$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{2}+0\right) - (0+0)\right] = \tfrac{\pi}{4}$$

**8.** $\int_0^{\pi/2}\sin^2(2\theta)\,d\theta = \int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}-0\right) - (0-0)\right] = \frac{\pi}{4}$

**9.**
$$\begin{aligned}\int_0^{\pi}\sin^4(3t)\,dt &= \int_0^{\pi}\left[\sin^2(3t)\right]^2 dt = \int_0^{\pi}\left[\tfrac{1}{2}(1-\cos 6t)\right]^2 dt = \tfrac{1}{4}\int_0^{\pi}(1 - 2\cos 6t + \cos^2 6t)\,dt \\ &= \tfrac{1}{4}\int_0^{\pi}\left[1 - 2\cos 6t + \tfrac{1}{2}(1+\cos 12t)\right] dt = \tfrac{1}{4}\int_0^{\pi}\left(\tfrac{3}{2} - 2\cos 6t + \tfrac{1}{2}\cos 12t\right) dt \\ &= \tfrac{1}{4}\left[\tfrac{3}{2}t - \tfrac{1}{3}\sin 6t + \tfrac{1}{24}\sin 12t\right]_0^{\pi} = \tfrac{1}{4}\left[\left(\tfrac{3\pi}{2} - 0 + 0\right) - (0 - 0 + 0)\right] = \tfrac{3\pi}{8}\end{aligned}$$

**10.**
$$\begin{aligned}\int_0^{\pi}\cos^6\theta\,d\theta &= \int_0^{\pi}(\cos^2\theta)^3\,d\theta = \int_0^{\pi}\left[\tfrac{1}{2}(1+\cos 2\theta)\right]^3 d\theta = \tfrac{1}{8}\int_0^{\pi}(1 + 3\cos 2\theta + 3\cos^2 2\theta + \cos^3 2\theta)\,d\theta \\ &= \tfrac{1}{8}\left[\theta + \tfrac{3}{2}\sin 2\theta\right]_0^{\pi} + \tfrac{1}{8}\int_0^{\pi}\left[\tfrac{3}{2}(1+\cos 4\theta)\right] d\theta + \tfrac{1}{8}\int_0^{\pi}\left[(1-\sin^2 2\theta)\cos 2\theta\right] d\theta \\ &= \tfrac{1}{8}\pi + \tfrac{3}{16}\left[\theta + \tfrac{1}{4}\sin 4\theta\right]_0^{\pi} + \tfrac{1}{8}\int_0^{0}(1-u^2)\left(\tfrac{1}{2}du\right) \qquad [\,u = \sin 2\theta,\ du = 2\cos 2\theta\,d\theta] \\ &= \tfrac{\pi}{8} + \tfrac{3\pi}{16} + 0 = \tfrac{5\pi}{16}\end{aligned}$$

**11.**
$$\begin{aligned}\int (1+\cos\theta)^2\,d\theta &= \int (1 + 2\cos\theta + \cos^2\theta)\,d\theta = \theta + 2\sin\theta + \tfrac{1}{2}\int (1+\cos 2\theta)\,d\theta \\ &= \theta + 2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C = \tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta + C\end{aligned}$$

**12.** Let $u = x$, $dv = \cos^2 x\,dx$ $\Rightarrow$ $du = dx$, $v = \int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos 2x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$, so

$$\begin{aligned}\int x\cos^2 x\,dx &= x\left(\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right) - \int \left(\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right) dx = \tfrac{1}{2}x^2 + \tfrac{1}{4}x\sin 2x - \tfrac{1}{4}x^2 + \tfrac{1}{8}\cos 2x + C \\ &= \tfrac{1}{4}x^2 + \tfrac{1}{4}x\sin 2x + \tfrac{1}{8}\cos 2x + C\end{aligned}$$

**13.**
$$\begin{aligned}\int_0^{\pi/2}\sin^2 x\,\cos^2 x\,dx &= \int_0^{\pi/2}\tfrac{1}{4}(4\sin^2 x\,\cos^2 x)\,dx = \int_0^{\pi/2}\tfrac{1}{4}(2\sin x\,\cos x)^2 dx = \tfrac{1}{4}\int_0^{\pi/2}\sin^2 2x\,dx \\ &= \tfrac{1}{4}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 4x)\,dx = \tfrac{1}{8}\int_0^{\pi/2}(1-\cos 4x)\,dx = \tfrac{1}{8}\left[x - \tfrac{1}{4}\sin 4x\right]_0^{\pi/2} = \tfrac{1}{8}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{16}\end{aligned}$$

**14.** $\int_0^\pi \sin^2 t \cos^4 t\,dt = \frac{1}{4}\int_0^\pi (4\sin^2 t\cos^2 t)\cos^2 t\,dt = \frac{1}{4}\int_0^\pi (2\sin t\cos t)^2\,\frac{1}{2}(1+\cos 2t)\,dt$

$= \frac{1}{8}\int_0^\pi (\sin 2t)^2(1+\cos 2t)\,dt = \frac{1}{8}\int_0^\pi (\sin^2 2t + \sin^2 2t\cos 2t)\,dt$

$= \frac{1}{8}\int_0^\pi \sin^2 2t\,dt + \frac{1}{8}\int_0^\pi \sin^2 2t\cos 2t\,dt = \frac{1}{8}\int_0^\pi \frac{1}{2}(1-\cos 4t)\,dt + \frac{1}{8}\left[\frac{1}{3}\cdot\frac{1}{2}\sin^3 2t\right]_0^\pi$

$= \frac{1}{16}\left[t - \frac{1}{4}\sin 4t\right]_0^\pi + \frac{1}{8}(0-0) = \frac{1}{16}\left[(\pi-0)-0\right] = \frac{\pi}{16}$

**15.** $\displaystyle\int \frac{\cos^5\alpha}{\sqrt{\sin\alpha}}\,d\alpha = \int \frac{\cos^4\alpha}{\sqrt{\sin\alpha}}\cos\alpha\,d\alpha = \int \frac{\left(1-\sin^2\alpha\right)^2}{\sqrt{\sin\alpha}}\cos\alpha\,d\alpha \overset{s}{=} \int \frac{(1-u^2)^2}{\sqrt{u}}\,du$

$\displaystyle = \int \frac{1-2u^2+u^4}{u^{1/2}}\,du = \int (u^{-1/2} - 2u^{3/2} + u^{7/2})\,du = 2u^{1/2} - \frac{4}{5}u^{5/2} + \frac{2}{9}u^{9/2} + C$

$= \frac{2}{45}u^{1/2}(45 - 18u^2 + 5u^4) + C = \frac{2}{45}\sqrt{\sin\alpha}\,(45 - 18\sin^2\alpha + 5\sin^4\alpha) + C$

**16.** Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$ and

$$\int \cos\theta\,\cos^5(\sin\theta)\,d\theta = \int \cos^5 u\,du = \int(\cos^2 u)^2\cos u\,du = \int(1-\sin^2 u)^2\cos u\,du$$
$$= \int(1 - 2\sin^2 u + \sin^4 u)\cos u\,du = I$$

Now let $x = \sin u$. Then $dx = \cos u\,du$ and

$$I = \int(1-2x^2+x^4)\,dx = x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5 + C = \sin u - \tfrac{2}{3}\sin^3 u + \tfrac{1}{5}\sin^5 u + C$$
$$= \sin(\sin\theta) - \tfrac{2}{3}\sin^3(\sin\theta) + \tfrac{1}{5}\sin^5(\sin\theta) + C$$

**17.** $\displaystyle\int \cos^2 x\tan^3 x\,dx = \int \frac{\sin^3 x}{\cos x}\,dx \overset{c}{=} \int \frac{(1-u^2)(-du)}{u} = \int\left[\frac{-1}{u} + u\right]du$

$= -\ln|u| + \frac{1}{2}u^2 + C = \frac{1}{2}\cos^2 x - \ln|\cos x| + C$

**18.** $\displaystyle\int \cot^5\theta\sin^4\theta\,d\theta = \int \frac{\cos^5\theta}{\sin^5\theta}\sin^4\theta\,d\theta = \int\frac{\cos^5\theta}{\sin\theta}\,d\theta = \int\frac{\cos^4\theta}{\sin\theta}\cos\theta\,d\theta = \int\frac{\left(1-\sin^2\theta\right)^2}{\sin\theta}\cos\theta\,d\theta$

$\displaystyle\overset{s}{=}\int\frac{\left(1-u^2\right)^2}{u}\,du = \int\frac{1-2u^2+u^4}{u}\,du = \int\left(\frac{1}{u} - 2u + u^3\right)du$

$= \ln|u| - u^2 + \frac{1}{4}u^4 + C = \ln|\sin\theta| - \sin^2\theta + \frac{1}{4}\sin^4\theta + C$

**19.** $\displaystyle\int\frac{\cos x + \sin 2x}{\sin x}\,dx = \int\frac{\cos x + 2\sin x\cos x}{\sin x}\,dx = \int\frac{\cos x}{\sin x}\,dx + \int 2\cos x\,dx \overset{s}{=} \int\frac{1}{u}\,du + 2\sin x$

$= \ln|u| + 2\sin x + C = \ln|\sin x| + 2\sin x + C$

*Or*: Use the formula $\int\cot x\,dx = \ln|\sin x| + C$.

**20.** $\int\cos^2 x\sin 2x\,dx = 2\int\cos^3 x\sin x\,dx \overset{c}{=} -2\int u^3\,du = -\frac{1}{2}u^4 + C = -\frac{1}{2}\cos^4 x + C$

**21.** Let $u = \tan x$, $du = \sec^2 x\,dx$. Then $\int\sec^2 x\tan x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C$.

*Or:* Let $v = \sec x$, $dv = \sec x\tan x\,dx$. Then $\int\sec^2 x\tan x\,dx = \int v\,dv = \frac{1}{2}v^2 + C = \frac{1}{2}\sec^2 x + C$.

**22.** $\int_0^{\pi/2}\sec^4(t/2)\,dt = \int_0^{\pi/4}\sec^4 x\,(2\,dx)\quad\left[x = t/2,\ dx = \frac{1}{2}\,dt\right]\quad = 2\int_0^{\pi/4}\sec^2 x\,(1+\tan^2 x)\,dx$

$= 2\int_0^1(1+u^2)\,du\quad[u = \tan x,\ du = \sec^2 x\,dx]\quad = 2\left[u + \frac{1}{3}u^3\right]_0^1 = 2\left(1+\frac{1}{3}\right) = \frac{8}{3}$

**23.** $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

**24.** $\int (\tan^2 x + \tan^4 x)\,dx = \int \tan^2 x\,(1 + \tan^2 x)\,dx = \int \tan^2 x \sec^2 x\,dx = \int u^2\,du \quad [u = \tan x,\ du = \sec^2 x\,dx]$

$= \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$

**25.** $\int \sec^6 t\,dt = \int \sec^4 t \cdot \sec^2 t\,dt = \int (\tan^2 t + 1)^2 \sec^2 t\,dt = \int (u^2 + 1)^2\,du \quad [u = \tan t,\ du = \sec^2 t\,dt]$

$= \int (u^4 + 2u^2 + 1)\,du = \frac{1}{5}u^5 + \frac{2}{3}u^3 + u + C = \frac{1}{5}\tan^5 t + \frac{2}{3}\tan^3 t + \tan t + C$

**26.** $\int_0^{\pi/4} \sec^4\theta \tan^4\theta\,d\theta = \int_0^{\pi/4} (\tan^2\theta + 1)\tan^4\theta \sec^2\theta\,d\theta = \int_0^1 (u^2 + 1)u^4\,du \quad [u = \tan\theta,\ du = \sec^2\theta\,d\theta]$

$= \int_0^1 (u^6 + u^4)\,du = \left[\frac{1}{7}u^7 + \frac{1}{5}u^5\right]_0^1 = \frac{1}{7} + \frac{1}{5} = \frac{12}{35}$

**27.** $\int_0^{\pi/3} \tan^5 x \sec^4 x\,dx = \int_0^{\pi/3} \tan^5 x\,(\tan^2 x + 1)\sec^2 x\,dx = \int_0^{\sqrt{3}} u^5(u^2 + 1)\,du \quad [u = \tan x,\ du = \sec^2 x\,dx]$

$= \int_0^{\sqrt{3}} (u^7 + u^5)\,du = \left[\frac{1}{8}u^8 + \frac{1}{6}u^6\right]_0^{\sqrt{3}} = \frac{81}{8} + \frac{27}{6} = \frac{81}{8} + \frac{9}{2} = \frac{81}{8} + \frac{36}{8} = \frac{117}{8}$

*Alternate solution:*

$\int_0^{\pi/3} \tan^5 x \sec^4 x\,dx = \int_0^{\pi/3} \tan^4 x \sec^3 x \sec x \tan x\,dx = \int_0^{\pi/3} (\sec^2 x - 1)^2 \sec^3 x \sec x \tan x\,dx$

$= \int_1^2 (u^2 - 1)^2 u^3\,du \quad [u = \sec x,\ du = \sec x \tan x\,dx] \quad = \int_1^2 (u^4 - 2u^2 + 1)u^3\,du$

$= \int_1^2 (u^7 - 2u^5 + u^3)\,du = \left[\frac{1}{8}u^8 - \frac{1}{3}u^6 + \frac{1}{4}u^4\right]_1^2 = \left(32 - \frac{64}{3} + 4\right) - \left(\frac{1}{8} - \frac{1}{3} + \frac{1}{4}\right) = \frac{117}{8}$

**28.** $\int \tan^3(2x) \sec^5(2x)\,dx = \int \tan^2(2x) \sec^4(2x) \cdot \sec(2x) \tan(2x)\,dx$

$= \int (u^2 - 1)u^4\left(\frac{1}{2}\,du\right) \quad [u = \sec(2x),\ du = 2\sec(2x)\tan(2x)\,dx]$

$= \frac{1}{2}\int (u^6 - u^4)\,du = \frac{1}{14}u^7 - \frac{1}{10}u^5 + C = \frac{1}{14}\sec^7(2x) - \frac{1}{10}\sec^5(2x) + C$

**29.** $\int \tan^3 x \sec x\,dx = \int \tan^2 x \sec x \tan x\,dx = \int (\sec^2 x - 1) \sec x \tan x\,dx$

$= \int (u^2 - 1)\,du \quad [u = \sec x,\ du = \sec x \tan x\,dx] \quad = \frac{1}{3}u^3 - u + C = \frac{1}{3}\sec^3 x - \sec x + C$

**30.** $\int_0^{\pi/3} \tan^5 x \sec^6 x\,dx = \int_0^{\pi/3} \tan^5 x \sec^4 x \sec^2 x\,dx = \int_0^{\pi/3} \tan^5 x\,(1 + \tan^2 x)^2 \sec^2 x\,dx$

$= \int_0^{\sqrt{3}} u^5(1 + u^2)^2\,du \quad [u = \tan x,\ du = \sec^2 x\,dx] \quad = \int_0^{\sqrt{3}} u^5(1 + 2u^2 + u^4)\,du$

$= \int_0^{\sqrt{3}} (u^5 + 2u^7 + u^9)\,du = \left[\frac{1}{6}u^6 + \frac{1}{4}u^8 + \frac{1}{10}u^{10}\right]_0^{\sqrt{3}} = \frac{27}{6} + \frac{81}{4} + \frac{243}{10} = \frac{981}{20}$

*Alternate solution:*

$\int_0^{\pi/3} \tan^5 x \sec^6 x\,dx = \int_0^{\pi/3} \tan^4 x \sec^5 x \sec x \tan x\,dx = \int_0^{\pi/3} (\sec^2 x - 1)^2 \sec^5 x \sec x \tan x\,dx$

$= \int_1^2 (u^2 - 1)^2 u^5\,du \quad [u = \sec x,\ du = \sec x \tan x\,dx]$

$= \int_1^2 (u^4 - 2u^2 + 1)u^5\,du = \int_1^2 (u^9 - 2u^7 + u^5)\,du$

$= \left[\frac{1}{10}u^{10} - \frac{1}{4}u^8 + \frac{1}{6}u^6\right]_1^2 = \left(\frac{512}{5} - 64 + \frac{32}{3}\right) - \left(\frac{1}{10} - \frac{1}{4} + \frac{1}{6}\right) = \frac{981}{20}$

**31.** $\int \tan^5 x\,dx = \int (\sec^2 x - 1)^2 \tan x\,dx = \int \sec^4 x \tan x\,dx - 2\int \sec^2 x \tan x\,dx + \int \tan x\,dx$

$= \int \sec^3 x \sec x \tan x\,dx - 2\int \tan x \sec^2 x\,dx + \int \tan x\,dx$

$= \frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C \quad [\text{or } \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C]$

**32.** $\int \tan^6 ay\,dy = \int \tan^4 ay\,(\sec^2 ay - 1)\,dy = \int \tan^4 ay\,\sec^2 ay\,dy - \int \tan^4 ay\,dy$

$= \frac{1}{5a}\tan^5 ay - \int \tan^2 ay\,(\sec^2 ay - 1)\,dy = \frac{1}{5a}\tan^5 ay - \int \tan^2 ay\,\sec^2 ay\,dy + \int(\sec^2 ay - 1)\,dy$

$= \frac{1}{5a}\tan^5 ay - \frac{1}{3a}\tan^3 ay + \frac{1}{a}\tan ay - y + C$

**33.** $\displaystyle\int \frac{\tan^3\theta}{\cos^4\theta}\,d\theta = \int \tan^3\theta\,\sec^4\theta\,d\theta = \int \tan^3\theta\cdot(\tan^2\theta + 1)\cdot\sec^2\theta\,d\theta$

$= \int u^3(u^2+1)\,du \qquad [u = \tan\theta,\ du = \sec^2\theta\,d\theta]$

$= \int(u^5 + u^3)\,du = \frac{1}{6}u^6 + \frac{1}{4}u^4 + C = \frac{1}{6}\tan^6\theta + \frac{1}{4}\tan^4\theta + C$

**34.** $\int \tan^2 x\,\sec x\,dx = \int(\sec^2 x - 1)\,\sec x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx$

$= \frac{1}{2}(\sec x\,\tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| + C$ [by Example 8 and (1)]

$= \frac{1}{2}(\sec x\,\tan x - \ln|\sec x + \tan x|) + C$

**35.** Let $u = x, dv = \sec x\,\tan x\,dx \Rightarrow du = dx, v = \sec x$. Then

$\int x\,\sec x\,\tan x\,dx = x\sec x - \int \sec x\,dx = x\sec x - \ln|\sec x + \tan x| + C.$

**36.** $\displaystyle\int \frac{\sin\phi}{\cos^3\phi}\,d\phi = \int \frac{\sin\phi}{\cos\phi}\cdot\frac{1}{\cos^2\phi}\,d\phi = \int \tan\phi\,\sec^2\phi\,d\phi = \int u\,du \qquad [u = \tan\phi,\ du = \sec^2\phi\,d\phi]$

$= \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2\phi + C$

*Alternate solution*: Let $u = \cos\phi$ to get $\frac{1}{2}\sec^2\phi + C$.

**37.** $\int_{\pi/6}^{\pi/2}\cot^2 x\,dx = \int_{\pi/6}^{\pi/2}(\csc^2 x - 1)\,dx = \left[-\cot x - x\right]_{\pi/6}^{\pi/2} = \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{\pi}{3}$

**38.** $\displaystyle\int_{\pi/4}^{\pi/2}\cot^3 x\,dx = \int_{\pi/4}^{\pi/2}\cot x\,(\csc^2 x - 1)\,dx = \int_{\pi/4}^{\pi/2}\cot x\,\csc^2 x\,dx - \int_{\pi/4}^{\pi/2}\frac{\cos x}{\sin x}\,dx$

$\displaystyle = \left[-\tfrac{1}{2}\cot^2 x - \ln|\sin x|\right]_{\pi/4}^{\pi/2} = (0 - \ln 1) - \left[-\tfrac{1}{2} - \ln\tfrac{1}{\sqrt{2}}\right] = \tfrac{1}{2} + \ln\tfrac{1}{\sqrt{2}} = \tfrac{1}{2}(1 - \ln 2)$

**39.** $\int \cot^3\alpha\,\csc^3\alpha\,d\alpha = \int \cot^2\alpha\,\csc^2\alpha\cdot\csc\alpha\,\cot\alpha\,d\alpha = \int(\csc^2\alpha - 1)\csc^2\alpha\cdot\csc\alpha\,\cot\alpha\,d\alpha$

$= \int(u^2 - 1)u^2\cdot(-du) \qquad [u = \csc\alpha,\ du = -\csc\alpha\cot\alpha\,d\alpha]$

$= \int(u^2 - u^4)\,du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \frac{1}{3}\csc^3\alpha - \frac{1}{5}\csc^5\alpha + C$

**40.** $\int \csc^4 x\,\cot^6 x\,dx = \int \cot^6 x\,(\cot^2 x + 1)\,\csc^2 x\,dx = \int u^6(u^2+1)\cdot(-du) \qquad [u = \cot x,\ du = -\csc^2 x\,dx]$

$= \int u^6(u^2+1)\cdot(-du) \qquad [u = \cot x,\ du = -\csc^2 x\,dx]$

$= \int(-u^8 - u^6)\,du = -\frac{1}{9}u^9 - \frac{1}{7}u^7 + C = -\frac{1}{9}\cot^9 x - \frac{1}{7}\cot^7 x + C$

**41.** $\displaystyle I = \int \csc x\,dx = \int \frac{\csc x\,(\csc x - \cot x)}{\csc x - \cot x}\,dx = \int \frac{-\csc x\,\cot x + \csc^2 x}{\csc x - \cot x}\,dx$. Let $u = \csc x - \cot x \Rightarrow$

$du = (-\csc x\,\cot x + \csc^2 x)\,dx$. Then $I = \int du/u = \ln|u| = \ln|\csc x - \cot x| + C$.

**42.** Let $u = \csc x$, $dv = \csc^2 x\,dx$. Then $du = -\csc x\,\cot x\,dx$, $v = -\cot x \Rightarrow$

$$\int \csc^3 x\,dx = -\csc x\,\cot x - \int \csc x\,\cot^2 x\,dx = -\csc x\,\cot x - \int \csc x\,(\csc^2 x - 1)\,dx$$
$$= -\csc x\,\cot x + \int \csc x\,dx - \int \csc^3 x\,dx$$

Solving for $\int \csc^3 x\,dx$ and using Exercise 41, we get

$\int \csc^3 x\,dx = -\frac{1}{2}\csc x\,\cot x + \frac{1}{2}\int \csc x\,dx = -\frac{1}{2}\csc x\,\cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$. Thus,

$$\int_{\pi/6}^{\pi/3} \csc^3 x\,dx = \left[-\tfrac{1}{2}\csc x\,\cot x + \tfrac{1}{2}\ln|\csc x - \cot x|\right]_{\pi/6}^{\pi/3}$$
$$= -\tfrac{1}{2}\cdot\tfrac{2}{\sqrt{3}}\cdot\tfrac{1}{\sqrt{3}} + \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}} - \tfrac{1}{\sqrt{3}}\right| + \tfrac{1}{2}\cdot 2\cdot\sqrt{3} - \tfrac{1}{2}\ln\left|2 - \sqrt{3}\right|$$
$$= -\tfrac{1}{3} + \sqrt{3} + \tfrac{1}{2}\ln\tfrac{1}{\sqrt{3}} - \tfrac{1}{2}\ln\left(2 - \sqrt{3}\right) \approx 1.7825$$

**43.** $\int \sin 8x\,\cos 5x\,dx \overset{2a}{=} \int \frac{1}{2}[\sin(8x - 5x) + \sin(8x + 5x)]\,dx = \frac{1}{2}\int \sin 3x\,dx + \frac{1}{2}\int \sin 13x\,dx$

$$= -\tfrac{1}{6}\cos 3x - \tfrac{1}{26}\cos 13x + C$$

**44.** $\int \cos \pi x\,\cos 4\pi x\,dx \overset{2c}{=} \int \frac{1}{2}[\cos(\pi x - 4\pi x) + \cos(\pi x + 4\pi x)]\,dx = \frac{1}{2}\int \cos(-3\pi x)\,dx + \frac{1}{2}\int \cos(5\pi x)\,dx$

$$= \tfrac{1}{2}\int \cos 3\pi x\,dx + \tfrac{1}{2}\int \cos 5\pi x\,dx = \tfrac{1}{6\pi}\sin 3\pi x + \tfrac{1}{10\pi}\sin 5\pi x + C$$

**45.** $\int \sin 5\theta\,\sin\theta\,d\theta \overset{2b}{=} \int \frac{1}{2}[\cos(5\theta - \theta) - \cos(5\theta + \theta)]\,d\theta = \frac{1}{2}\int \cos 4\theta\,d\theta - \frac{1}{2}\int \cos 6\theta\,d\theta = \frac{1}{8}\sin 4\theta - \frac{1}{12}\sin 6\theta + C$

**46.** $\displaystyle\int \frac{\cos x + \sin x}{\sin 2x}\,dx = \frac{1}{2}\int \frac{\cos x + \sin x}{\sin x\,\cos x}\,dx = \frac{1}{2}\int (\csc x + \sec x)\,dx$

$$= \tfrac{1}{2}\Big(\ln|\csc x - \cot x| + \ln|\sec x + \tan x|\Big) + C \qquad \text{[by Exercise 41 and (1)]}$$

**47.** $\displaystyle\int \frac{1 - \tan^2 x}{\sec^2 x}\,dx = \int \left(\cos^2 x - \sin^2 x\right)dx = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$

**48.** $\displaystyle\int \frac{dx}{\cos x - 1} = \int \frac{1}{\cos x - 1}\cdot\frac{\cos x + 1}{\cos x + 1}\,dx = \int \frac{\cos x + 1}{\cos^2 x - 1}\,dx = \int \frac{\cos x + 1}{-\sin^2 x}\,dx$

$$= \int \left(-\cot x\,\csc x - \csc^2 x\right)dx = \csc x + \cot x + C$$

**49.** Let $u = \tan(t^2) \Rightarrow du = 2t\sec^2(t^2)\,dt$. Then $\int t\sec^2(t^2)\,\tan^4(t^2)\,dt = \int u^4\left(\frac{1}{2}\,du\right) = \frac{1}{10}u^5 + C = \frac{1}{10}\tan^5(t^2) + C$.

**50.** Let $u = \tan^7 x$, $dv = \sec x\,\tan x\,dx \Rightarrow du = 7\tan^6 x\,\sec^2 x\,dx$, $v = \sec x$. Then

$$\int \tan^8 x\,\sec x\,dx = \int \tan^7 x\cdot\sec x\,\tan x\,dx = \tan^7 x\,\sec x - \int 7\tan^6 x\sec^2 x\sec x\,dx$$
$$= \tan^7 x\,\sec x - 7\int \tan^6 x\,(\tan^2 x + 1)\,\sec x\,dx$$
$$= \tan^7 x\,\sec x - 7\int \tan^8 x\sec x\,dx - 7\int \tan^6 x\sec x\,dx.$$

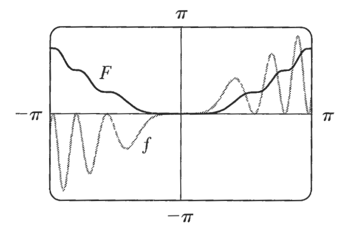
Thus, $8\int \tan^8 x\sec x\,dx = \tan^7 x\,\sec x - 7\int \tan^6 x\sec x\,dx$ and

$$\int_0^{\pi/4} \tan^8 x\sec x\,dx = \frac{1}{8}\left[\tan^7 x\,\sec x\right]_0^{\pi/4} - \frac{7}{8}\int_0^{\pi/4} \tan^6 x\,\sec x\,dx = \frac{\sqrt{2}}{8} - \frac{7}{8}I.$$

In Exercises 51–54, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

**51.** Let $u = x^2$, so that $du = 2x\,dx$. Then

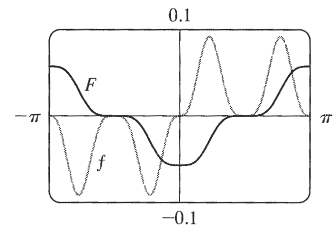
$$\int x\sin^2(x^2)\,dx = \int \sin^2 u\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int \tfrac{1}{2}(1-\cos 2u)\,du$$
$$= \tfrac{1}{4}\left(u - \tfrac{1}{2}\sin 2u\right) + C = \tfrac{1}{4}u - \tfrac{1}{4}\left(\tfrac{1}{2}\cdot 2\sin u\,\cos u\right) + C$$
$$= \tfrac{1}{4}x^2 - \tfrac{1}{4}\sin(x^2)\,\cos(x^2) + C$$




We see from the graph that this is reasonable, since $F$ increases where $f$ is positive and $F$ decreases where $f$ is negative. Note also that $f$ is an odd function and $F$ is an even function.

**52.** $\int \sin^3 x\,\cos^4 x\,dx = \int \cos^4 x\,(1-\cos^2 x)\sin x\,dx$

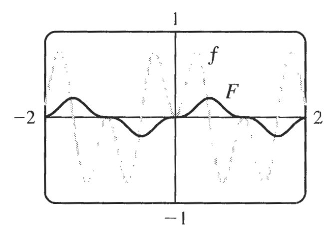
$$\overset{c}{=} \int u^4(1-u^2)(-du) = \int (u^6 - u^4)\,du$$
$$= \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C = \tfrac{1}{7}\cos^7 x - \tfrac{1}{5}\cos^5 x + C$$




We see from the graph that this is reasonable, since $F$ increases where $f$ is positive and $F$ decreases where $f$ is negative. Note also that $f$ is an odd function and $F$ is an even function.

**53.** $\int \sin 3x\,\sin 6x\,dx = \int \tfrac{1}{2}[\cos(3x-6x) - \cos(3x+6x)]\,dx$

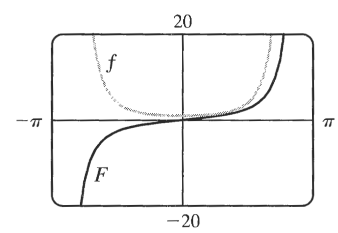
$$= \tfrac{1}{2}\int(\cos 3x - \cos 9x)\,dx$$
$$= \tfrac{1}{6}\sin 3x - \tfrac{1}{18}\sin 9x + C$$




Notice that $f(x) = 0$ whenever $F$ has a horizontal tangent.

**54.** $\int \sec^4 \tfrac{x}{2}\,dx = \int \left(\tan^2 \tfrac{x}{2} + 1\right)\sec^2 \tfrac{x}{2}\,dx$

$$= \int (u^2+1)\,2\,du \qquad \left[u = \tan\tfrac{x}{2},\ du = \tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx\right]$$
$$= \tfrac{2}{3}u^3 + 2u + C = \tfrac{2}{3}\tan^3\tfrac{x}{2} + 2\tan\tfrac{x}{2} + C$$




Notice that $F$ is increasing and $f$ is positive on the intervals on which they are defined. Also, $F$ has no horizontal tangent and $f$ is never zero.

**55.** $f_{\text{ave}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x\,\cos^3 x\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 x\,(1-\sin^2 x)\,\cos x\,dx$

$$= \tfrac{1}{2\pi}\int_0^0 u^2(1-u^2)\,du \qquad [\text{where } u = \sin x]$$
$$= 0$$

**56.** (a) Let $u = \cos x$. Then $du = -\sin x\,dx \;\Rightarrow\; \int \sin x\,\cos x\,dx = \int u(-du) = -\tfrac{1}{2}u^2 + C = -\tfrac{1}{2}\cos^2 x + C_1$.

(b) Let $u = \sin x$. Then $du = \cos x\,dx \;\Rightarrow\; \int \sin x\,\cos x\,dx = \int u\,du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\sin^2 x + C_2$.

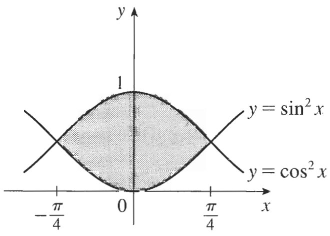
(c) $\int \sin x\,\cos x\,dx = \int \tfrac{1}{2}\sin 2x\,dx = -\tfrac{1}{4}\cos 2x + C_3$

(d) Let $u = \sin x$, $dv = \cos x\,dx$. Then $du = \cos x\,dx$, $v = \sin x$, so $\int \sin x\,\cos x\,dx = \sin^2 x - \int \sin x\,\cos x\,dx$, by Equation 7.1.2, so $\int \sin x\,\cos x\,dx = \tfrac{1}{2}\sin^2 x + C_4$.

**57.** $A = \int_{-\pi/4}^{\pi/4} (\cos^2 x - \sin^2 x)\, dx = \int_{-\pi/4}^{\pi/4} \cos 2x\, dx$

$= 2\int_0^{\pi/4} \cos 2x\, dx = 2\left[\frac{1}{2}\sin 2x\right]_0^{\pi/4} = \left[\sin 2x\right]_0^{\pi/4}$

$= 1 - 0 = 1$



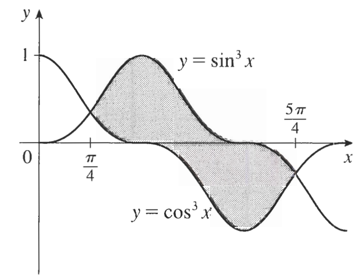

**58.** $A = \int_{\pi/4}^{5\pi/4} (\sin^3 x - \cos^3 x)\, dx = \int_{\pi/4}^{5\pi/4} \sin^3 x\, dx - \int_{\pi/4}^{5\pi/4} \cos^3 x\, dx$

$= \int_{\pi/4}^{5\pi/4} (1 - \cos^2 x)\, \sin x\, dx - \int_{\pi/4}^{5\pi/4} (1 - \sin^2 x)\, \cos x\, dx$

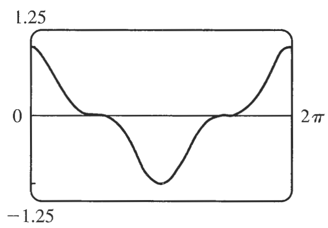
$\overset{\text{c,s}}{=} \int_{\sqrt{2}/2}^{-\sqrt{2}/2} (1 - u^2)(-du) - \int_{\sqrt{2}/2}^{-\sqrt{2}/2} (1 - u^2)\, du$

$= 2\int_0^{\sqrt{2}/2} (1 - u^2)\, du + 2\int_0^{\sqrt{2}/2} (1 - u^2)\, du = 4\left[u - \frac{1}{3}u^3\right]_0^{\sqrt{2}/2}$

$= 4\left[\left(\frac{\sqrt{2}}{2} - \frac{1}{3}\cdot\frac{\sqrt{2}}{4}\right) - 0\right] = 2\sqrt{2} - \frac{1}{3}\sqrt{2} = \frac{5}{3}\sqrt{2}$

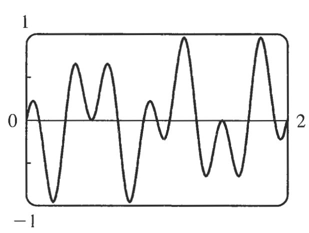



**59.**




It seems from the graph that $\int_0^{2\pi} \cos^3 x\, dx = 0$, since the area below the $x$-axis and above the graph looks about equal to the area above the axis and below the graph. By Example 1, the integral is $\left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{2\pi} = 0$. Note that due to symmetry, the integral of any odd power of $\sin x$ or $\cos x$ between limits which differ by $2n\pi$ ($n$ any integer) is 0.

**60.**




It seems from the graph that $\int_0^2 \sin 2\pi x\, \cos 5\pi x\, dx = 0$, since each bulge above the $x$-axis seems to have a corresponding depression below the $x$-axis. To evaluate the integral, we use a trigonometric identity:

$\int_0^1 \sin 2\pi x\, \cos 5\pi x\, dx = \frac{1}{2}\int_0^2 [\sin(2\pi x - 5\pi x) + \sin(2\pi x + 5\pi x)]\, dx$

$= \frac{1}{2}\int_0^2 [\sin(-3\pi x) + \sin 7\pi x]\, dx$

$= \frac{1}{2}\left[\frac{1}{3\pi}\cos(-3\pi x) - \frac{1}{7\pi}\cos 7\pi x\right]_0^2$

$= \frac{1}{2}\left[\frac{1}{3\pi}(1 - 1) - \frac{1}{7\pi}(1 - 1)\right] = 0$

**61.** Using disks, $V = \int_{\pi/2}^{\pi} \pi \sin^2 x\, dx = \pi\int_{\pi/2}^{\pi} \frac{1}{2}(1 - \cos 2x)\, dx = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\pi/2}^{\pi} = \pi\left(\frac{\pi}{2} - 0 - \frac{\pi}{4} + 0\right) = \frac{\pi^2}{4}$

**62.** Using disks,

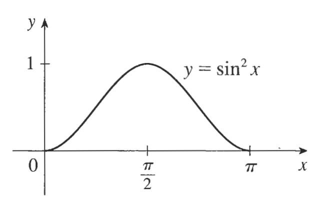
$V = \int_0^{\pi} \pi(\sin^2 x)^2\, dx = 2\pi\int_0^{\pi/2} \left[\frac{1}{2}(1 - \cos 2x)\right]^2 dx$

$= \frac{\pi}{2}\int_0^{\pi/2} (1 - 2\cos 2x + \cos^2 2x)\, dx$

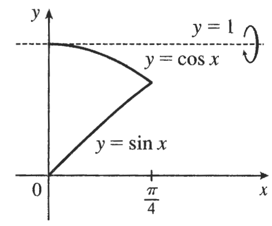
$= \frac{\pi}{2}\int_0^{\pi/2} \left[1 - 2\cos 2x + \frac{1}{2}(1 - \cos 4x)\right] dx$

$= \frac{\pi}{2}\int_0^{\pi/2} \left(\frac{3}{2} - 2\cos 2x - \frac{1}{2}\cos 4x\right) dx = \frac{\pi}{2}\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right]_0^{\pi/2}$

$= \frac{\pi}{2}\left[\left(\frac{3\pi}{4} - 0 + 0\right) - (0 - 0 + 0)\right] = \frac{3}{8}\pi^2$

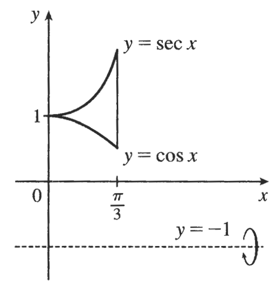


**63.** Using washers,




$$
\begin{aligned}
V &= \int_0^{\pi/4} \pi\left[(1-\sin x)^2 - (1-\cos x)^2\right] dx \\
&= \pi \int_0^{\pi/4} \left[(1 - 2\sin x + \sin^2 x) - (1 - 2\cos x + \cos^2 x)\right] dx \\
&= \pi \int_0^{\pi/4} (2\cos x - 2\sin x + \sin^2 x - \cos^2 x)\, dx \\
&= \pi \int_0^{\pi/4} (2\cos x - 2\sin x - \cos 2x)\, dx = \pi\left[2\sin x + 2\cos x - \tfrac{1}{2}\sin 2x\right]_0^{\pi/4} \\
&= \pi\left[\left(\sqrt{2} + \sqrt{2} - \tfrac{1}{2}\right) - (0 + 2 - 0)\right] = \pi\left(2\sqrt{2} - \tfrac{5}{2}\right)
\end{aligned}
$$

**64.** Using washers,




$$
\begin{aligned}
V &= \int_0^{\pi/3} \pi\left\{[\sec x - (-1)]^2 - [\cos x - (-1)]^2\right\} dx \\
&= \pi \int_0^{\pi/3} [(\sec^2 x + 2\sec x + 1) - (\cos^2 x + 2\cos x + 1)]\, dx \\
&= \pi \int_0^{\pi/3} \left[\sec^2 x + 2\sec x - \tfrac{1}{2}(1 + \cos 2x) - 2\cos x\right] dx \\
&= \pi\left[\tan x + 2\ln|\sec x + \tan x| - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x - 2\sin x\right]_0^{\pi/3} \\
&= \pi\left[\left(\sqrt{3} + 2\ln(2+\sqrt{3}) - \tfrac{\pi}{6} - \tfrac{1}{8}\sqrt{3} - \sqrt{3}\right) - 0\right] \\
&= 2\pi\ln\left(2+\sqrt{3}\right) - \tfrac{1}{6}\pi^2 - \tfrac{1}{8}\pi\sqrt{3}
\end{aligned}
$$

**65.** $s = f(t) = \int_0^t \sin\omega u \cos^2 \omega u\, du$. Let $y = \cos\omega u \;\Rightarrow\; dy = -\omega \sin\omega u\, du$. Then

$$s = -\tfrac{1}{\omega}\int_1^{\cos\omega t} y^2 dy = -\tfrac{1}{\omega}\left[\tfrac{1}{3}y^3\right]_1^{\cos\omega t} = \tfrac{1}{3\omega}(1 - \cos^3 \omega t).$$

**66.** (a) We want to calculate the square root of the average value of $[E(t)]^2 = [155\sin(120\pi t)]^2 = 155^2\sin^2(120\pi t)$. First, we calculate the average value itself, by integrating $[E(t)]^2$ over one cycle (between $t = 0$ and $t = \frac{1}{60}$, since there are 60 cycles per second) and dividing by $\left(\frac{1}{60} - 0\right)$:

$$
\begin{aligned}
[E(t)]^2_{\text{ave}} &= \tfrac{1}{1/60}\int_0^{1/60}[155^2\sin^2(120\pi t)]\, dt = 60\cdot 155^2 \int_0^{1/60} \tfrac{1}{2}[1 - \cos(240\pi t)]\, dt \\
&= 60\cdot 155^2\left(\tfrac{1}{2}\right)\left[t - \tfrac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 60\cdot 155^2\left(\tfrac{1}{2}\right)\left[\left(\tfrac{1}{60} - 0\right) - (0 - 0)\right] = \tfrac{155^2}{2}
\end{aligned}
$$

The RMS value is just the square root of this quantity, which is $\frac{155}{\sqrt{2}} \approx 110$ V.

(b) $220 = \sqrt{[E(t)]^2_{\text{ave}}} \;\Rightarrow$

$$
\begin{aligned}
220^2 = [E(t)]^2_{\text{ave}} &= \tfrac{1}{1/60}\int_0^{1/60} A^2\sin^2(120\pi t)\, dt = 60A^2\int_0^{1/60}\tfrac{1}{2}[1 - \cos(240\pi t)]\, dt \\
&= 30A^2\left[t - \tfrac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 30A^2\left[\left(\tfrac{1}{60} - 0\right) - (0 - 0)\right] = \tfrac{1}{2}A^2
\end{aligned}
$$

Thus, $220^2 = \frac{1}{2}A^2 \;\Rightarrow\; A = 220\sqrt{2} \approx 311$ V.

**67.** Just note that the integrand is odd $[f(-x) = -f(x)]$.

*Or:* If $m \neq n$, calculate

$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = \int_{-\pi}^{\pi} \tfrac{1}{2}[\sin(m-n)x + \sin(m+n)x]\, dx = \frac{1}{2}\left[-\frac{\cos(m-n)x}{m-n} - \frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$$

If $m = n$, then the first term in each set of brackets is zero.

**68.** $\int_{-\pi}^{\pi} \sin mx \, \sin nx \, dx = \int_{-\pi}^{\pi} \frac{1}{2}[\cos(m-n)x - \cos(m+n)x] \, dx$.

If $m \neq n$, this is equal to $\frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$.

If $m = n$, we get $\int_{-\pi}^{\pi} \frac{1}{2}[1 - \cos(m+n)x] \, dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} - \left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi - 0 = \pi$.

**69.** $\int_{-\pi}^{\pi} \cos mx \, \cos nx \, dx = \int_{-\pi}^{\pi} \frac{1}{2}[\cos(m-n)x + \cos(m+n)x] \, dx$.

If $m \neq n$, this is equal to $\frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$.

If $m = n$, we get $\int_{-\pi}^{\pi} \frac{1}{2}[1 + \cos(m+n)x] \, dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} + \left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi + 0 = \pi$.

**70.** $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\left(\sum_{n=1}^{m} a_n \sin nx\right)\sin mx\right] dx = \sum_{n=1}^{m} \frac{a_n}{\pi}\int_{-\pi}^{\pi} \sin mx \, \sin nx \, dx$. By Exercise 68, every term is zero except the $m$th one, and that term is $\frac{a_m}{\pi} \cdot \pi = a_m$.