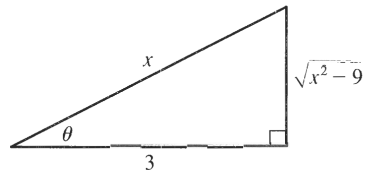
1. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$dx = 3\sec\theta\,\tan\theta\,d\theta$ and

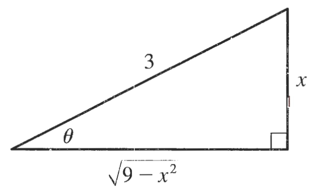


$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9(\sec^2\theta - 1)} = \sqrt{9\tan^2\theta}$$
$$= 3\,|\tan\theta| = 3\tan\theta \text{ for the relevant values of } \theta.$$

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{1}{9\sec^2\theta \cdot 3\tan\theta}\,3\sec\theta\,\tan\theta\,d\theta = \tfrac{1}{9}\int \cos\theta\,d\theta = \tfrac{1}{9}\sin\theta + C = \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

Note that $-\sec(\theta + \pi) = \sec\theta$, so the figure is sufficient for the case $\pi \le \theta < \frac{3\pi}{2}$.

2. Let $x = 3\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 3\cos\theta\,d\theta$ and

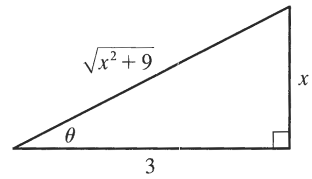


$$\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta}$$
$$= 3\,|\cos\theta| = 3\cos\theta \text{ for the relevant values of } \theta.$$

$$\int x^3\sqrt{9-x^2}\,dx = \int 3^3\sin^3\theta \cdot 3\cos\theta \cdot 3\cos\theta\,d\theta = 3^5\int \sin^3\theta\,\cos^2\theta\,d\theta = 3^5\int \sin^2\theta\,\cos^2\theta\,\sin\theta\,d\theta$$
$$= 3^5\int (1-\cos^2\theta)\,\cos^2\theta\,\sin\theta\,d\theta = 3^5\int (1-u^2)u^2\,(-du) \qquad [u = \cos\theta,\ du = -\sin\theta\,d\theta]$$
$$= 3^5\int (u^4 - u^2)\,du = 3^5\left(\tfrac{1}{5}u^5 - \tfrac{1}{3}u^3\right) + C = 3^5\left(\tfrac{1}{5}\cos^5\theta - \tfrac{1}{3}\cos^3\theta\right) + C$$
$$= 3^5\left[\frac{1}{5}\frac{\left(9-x^2\right)^{5/2}}{3^5} - \frac{1}{3}\frac{\left(9-x^2\right)^{3/2}}{3^3}\right] + C$$
$$= \tfrac{1}{5}(9-x^2)^{5/2} - 3(9-x^2)^{3/2} + C \quad \text{or} \quad -\tfrac{1}{5}(x^2+6)(9-x^2)^{3/2} + C$$

3. Let $x = 3\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 3\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = \sqrt{9(\tan^2\theta + 1)} = \sqrt{9\sec^2\theta}$$
$$= 3\,|\sec\theta| = 3\sec\theta \text{ for the relevant values of } \theta.$$



$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{3^3\tan^3\theta}{3\sec\theta}\,3\sec^2\theta\,d\theta = 3^3\int \tan^3\theta\,\sec\theta d\theta = 3^3\int \tan^2\theta\,\tan\theta\,\sec\theta\,d\theta$$
$$= 3^3\int(\sec^2\theta - 1)\,\tan\theta\,\sec\theta\,d\theta = 3^3\int(u^2-1)\,du \qquad [u = \sec\theta,\, du = \sec\theta\,\tan\theta\,d\theta]$$
$$= 3^3\left(\tfrac{1}{3}u^3 - u\right) + C = 3^3\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right) + C = 3^3\left[\frac{1}{3}\frac{(x^2+9)^{3/2}}{3^3} - \frac{\sqrt{x^2+9}}{3}\right] + C$$
$$= \tfrac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C \quad \text{or} \quad \tfrac{1}{3}(x^2-18)\sqrt{x^2+9} + C$$

4. Let $x = 4\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 4\cos\theta\,d\theta$ and

$\sqrt{16-x^2} = \sqrt{16 - 16\sin^2\theta} = \sqrt{16\cos^2\theta} = 4\,|\cos\theta| = 4\cos\theta$. When $x = 0$, $4\sin\theta = 0 \;\Rightarrow\; \theta = 0$,

and when $x = 2\sqrt{3}$, $4\sin\theta = 2\sqrt{3} \;\Rightarrow\; \sin\theta = \frac{\sqrt{3}}{2} \;\Rightarrow\; \theta = \frac{\pi}{3}$. Thus, substitution gives

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx = \int_0^{\pi/3} \frac{4^3\sin^3\theta}{4\cos\theta}\,4\cos\theta\,d\theta = 4^3\int_0^{\pi/3}\sin^3\theta\,d\theta = 4^3\int_0^{\pi/3}(1-\cos^2\theta)\,\sin\theta\,d\theta$$
$$\overset{c}{=} -4^3\int_1^{1/2}(1-u^2)\,du = -64\left[u - \tfrac{1}{3}u^3\right]_1^{1/2}$$
$$= -64\left[\left(\tfrac{1}{2} - \tfrac{1}{24}\right) - \left(1 - \tfrac{1}{3}\right)\right] = -64\left(-\tfrac{5}{24}\right) = \tfrac{40}{3}$$

Or: Let $u = 16 - x^2$, $x^2 = 16 - u$, $du = -2x\,dx$.

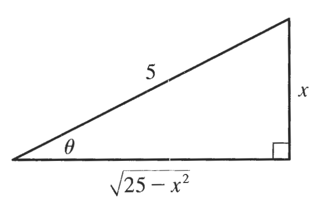
5. Let $t = \sec\theta$, so $dt = \sec\theta\,\tan\theta\,d\theta$, $t = \sqrt{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$, and $t = 2 \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta\,\tan\theta}\sec\theta\,\tan\theta\,d\theta = \int_{\pi/4}^{\pi/3}\frac{1}{\sec^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3}\cos^2\theta\,d\theta$$
$$= \int_{\pi/4}^{\pi/3}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$
$$= \tfrac{1}{2}\left[\left(\tfrac{\pi}{3} + \tfrac{1}{2}\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{\pi}{4} + \tfrac{1}{2}\cdot 1\right)\right] = \tfrac{1}{2}\left(\tfrac{\pi}{12} + \tfrac{\sqrt{3}}{4} - \tfrac{1}{2}\right) = \tfrac{\pi}{24} + \tfrac{\sqrt{3}}{8} - \tfrac{1}{4}$$

6. Let $x = \sec\theta$, so $dx = \sec\theta\,\tan\theta\,d\theta$, $x = 1 \;\Rightarrow\; \theta = 0$, and $x = 2 \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

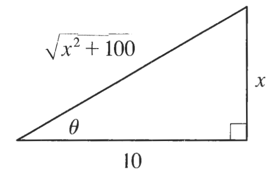
$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx = \int_0^{\pi/3}\frac{\tan\theta}{\sec\theta}\sec\theta\,\tan\theta\,d\theta = \int_0^{\pi/3}\tan^2\theta\,d\theta = \int_0^{\pi/3}(\sec^2\theta - 1)\,d\theta$$
$$= \left[\tan\theta - \theta\right]_0^{\pi/3} = \left(\sqrt{3} - \tfrac{\pi}{3}\right) - 0 = \sqrt{3} - \tfrac{\pi}{3}$$

7. Let $x = 5\sin\theta$, so $dx = 5\cos\theta\,d\theta$. Then



$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \int \frac{1}{5^2\sin^2\theta\cdot 5\cos\theta}\,5\cos\theta\,d\theta = \frac{1}{25}\int\csc^2\theta\,d\theta$$
$$= -\frac{1}{25}\cot\theta + C = -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C$$

8. Let $x = 10\tan\theta$, so $dx = 10\sec^2\theta\,d\theta$. Then



$$\int \frac{x^3}{\sqrt{x^2+100}}\,dx = \int \frac{1000\tan^3\theta}{10\sec\theta}\,10\sec^2\theta\,d\theta$$

$$= 1000\int \tan^3\theta\,\sec\theta\,d\theta = 1000\int(\sec^2\theta - 1)\,\sec\theta\,\tan\theta\,d\theta$$

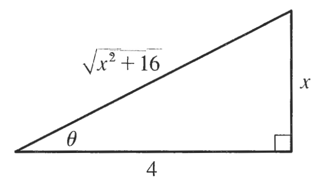
$$= 1000\int(u^2-1)\,du \qquad [u = \sec\theta,\ du = \sec\theta\,\tan\theta\,d\theta]$$

$$= 1000\left(\tfrac{1}{3}u^3 - u\right) + C = \tfrac{1000}{3}u(u^2-3) + C = \tfrac{1000}{3}\sec\theta\,(\sec^2\theta - 3) + C$$

$$= \frac{1000}{3}\,\frac{\sqrt{x^2+100}}{10}\left(\frac{x^2+100}{100} - 3\right) + C = \frac{100}{3}\sqrt{x^2+100}\,\frac{x^2-200}{100} + C$$

$$= \tfrac{1}{3}(x^2-200)\sqrt{x^2+100} + C$$

9. Let $x = 4\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 4\sec^2\theta\,d\theta$ and



$$\sqrt{x^2+16} = \sqrt{16\tan^2\theta + 16} = \sqrt{16(\tan^2\theta + 1)}$$

$$= \sqrt{16\sec^2\theta} = 4\,|\sec\theta|$$

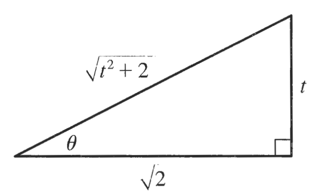
$= 4\sec\theta$ for the relevant values of θ.

$$\int \frac{dx}{\sqrt{x^2+16}} = \int \frac{4\sec^2\theta\,d\theta}{4\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C_1$$

$$= \ln\left|\sqrt{x^2+16} + x\right| - \ln|4| + C_1 = \ln\left(\sqrt{x^2+16} + x\right) + C, \text{ where } C = C_1 - \ln 4.$$

(Since $\sqrt{x^2+16} + x > 0$, we don't need the absolute value.)

10. Let $t = \sqrt{2}\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dt = \sqrt{2}\sec^2\theta\,d\theta$ and



$$\sqrt{t^2+2} = \sqrt{2\tan^2\theta + 2} = \sqrt{2(\tan^2\theta+1)} = \sqrt{2\sec^2\theta}$$

$= \sqrt{2}\,|\sec\theta| = \sqrt{2}\sec\theta$ for the relevant values of θ.

$$\int \frac{t^5}{\sqrt{t^2+2}}\,dt = \int \frac{4\sqrt{2}\tan^5\theta}{\sqrt{2}\sec\theta}\sqrt{2}\sec^2\theta\,d\theta = 4\sqrt{2}\int \tan^5\theta\sec\theta\,d\theta$$

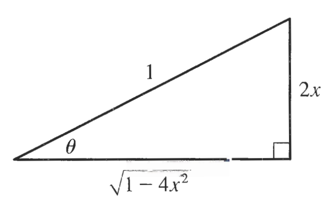
$$= 4\sqrt{2}\int(\sec^2\theta - 1)^2\sec\theta\,\tan\theta\,d\theta = 4\sqrt{2}\int(u^2-1)^2\,du \quad [u = \sec\theta,\ du = \sec\theta\,\tan\theta\,d\theta]$$

$$= 4\sqrt{2}\int(u^4 - 2u^2 + 1)\,du = 4\sqrt{2}\left(\tfrac{1}{5}u^5 - \tfrac{2}{3}u^3 + u\right) + C$$

$$= \frac{4\sqrt{2}}{15}u(3u^4 - 10u^2 + 15) + C = \frac{4\sqrt{2}}{15}\cdot\frac{\sqrt{t^2+2}}{\sqrt{2}}\left[3\cdot\frac{(t^2+2)^2}{2^2} - 10\,\frac{t^2+2}{2} + 15\right] + C$$

$$= \tfrac{4}{15}\sqrt{t^2+2}\cdot\tfrac{1}{4}\left[3(t^4 + 4t^2 + 4) - 20(t^2+2) + 60\right] + C = \tfrac{1}{15}\sqrt{t^2+2}\,(3t^4 - 8t^2 + 32) + C$$

11. Let $2x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $x = \frac{1}{2}\sin\theta$, $dx = \frac{1}{2}\cos\theta\,d\theta$, and $\sqrt{1-4x^2} = \sqrt{1-(2x)^2} = \cos\theta$.



$$\int\sqrt{1-4x^2}\,dx = \int\cos\theta\left(\tfrac{1}{2}\cos\theta\right)d\theta = \tfrac{1}{4}\int(1+\cos 2\theta)\,d\theta$$

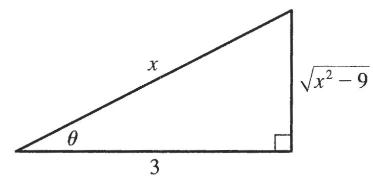
$$= \tfrac{1}{4}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{4}(\theta + \sin\theta\,\cos\theta) + C$$

$$= \tfrac{1}{4}\left[\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\,\right] + C$$

12. $\int_0^1 x\sqrt{x^2+4}\,dx = \int_4^5 \sqrt{u}\left(\frac{1}{2}\,du\right)\quad [u = x^2+4,\ du = 2x\,dx]\quad = \frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_4^5 = \frac{1}{3}\left(5\sqrt{5}-8\right)$

13. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$dx = 3\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$, so

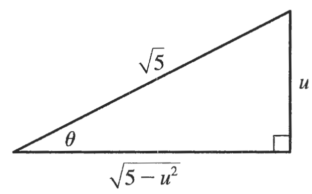


$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\,\tan\theta\,d\theta = \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \tfrac{1}{3}\int \sin^2\theta\,d\theta = \tfrac{1}{3}\int \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{6}\theta - \tfrac{1}{12}\sin 2\theta + C = \tfrac{1}{6}\theta - \tfrac{1}{6}\sin\theta\,\cos\theta + C$$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\frac{\sqrt{x^2-9}}{x}\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

14. Let $u = \sqrt{5}\sin\theta$, so $du = \sqrt{5}\cos\theta\,d\theta$. Then



$$\int \frac{du}{u\sqrt{5-u^2}} = \int \frac{1}{\sqrt{5}\sin\theta\cdot\sqrt{5}\cos\theta}\sqrt{5}\cos\theta\,d\theta = \frac{1}{\sqrt{5}}\int \csc\theta\,d\theta$$

$$= \tfrac{1}{\sqrt{5}}\ln\left|\csc\theta - \cot\theta\right| + C \qquad \text{[by Exercise 7.2.39]}$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{u} - \frac{\sqrt{5-u^2}}{u}\right| + C = \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}-\sqrt{5-u^2}}{u}\right| + C$$

15. Let $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$ and $x = a \;\Rightarrow\; \theta = \frac{\pi}{2}$. Then

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx = \int_0^{\pi/2} a^2\sin^2\theta\,(a\cos\theta)\,a\cos\theta\,d\theta = a^4\int_0^{\pi/2}\sin^2\theta\,\cos^2\theta\,d\theta = a^4\int_0^{\pi/2}\left[\tfrac{1}{2}(2\sin\theta\,\cos\theta)\right]^2 d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2}\sin^2 2\theta\,d\theta = \frac{a^4}{4}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)\,d\theta = \frac{a^4}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}$$

$$= \frac{a^4}{8}\left[\left(\frac{\pi}{2}-0\right)-0\right] = \frac{\pi}{16}a^4$$

16. Let $x = \frac{1}{3}\sec\theta$, so $dx = \frac{1}{3}\sec\theta\,\tan\theta\,d\theta$, $x = \sqrt{2}/3 \;\Rightarrow\; \theta = \frac{\pi}{4}$, $x = \frac{2}{3} \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

$$\int_{\sqrt{2}/3}^{2/3}\frac{dx}{x^5\sqrt{9x^2-1}} = \int_{\pi/4}^{\pi/3}\frac{\frac{1}{3}\sec\theta\,\tan\theta\,d\theta}{\left(\frac{1}{3}\right)^5\sec^5\theta\,\tan\theta} = 3^4\int_{\pi/4}^{\pi/3}\cos^4\theta\,d\theta = 81\int_{\pi/4}^{\pi/3}\left[\tfrac{1}{2}(1+\cos 2\theta)\right]^2 d\theta$$

$$= \tfrac{81}{4}\int_{\pi/4}^{\pi/3}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta = \tfrac{81}{4}\int_{\pi/4}^{\pi/3}\left[1+2\cos 2\theta+\tfrac{1}{2}(1+\cos 4\theta)\right]d\theta$$

$$= \tfrac{81}{4}\int_{\pi/4}^{\pi/3}\left(\tfrac{3}{2}+2\cos 2\theta+\tfrac{1}{2}\cos 4\theta\right)d\theta = \tfrac{81}{4}\left[\tfrac{3}{2}\theta+\sin 2\theta+\tfrac{1}{8}\sin 4\theta\right]_{\pi/4}^{\pi/3}$$

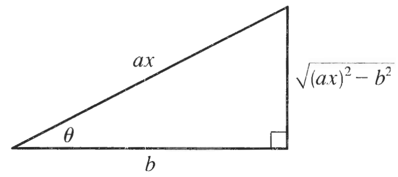
$$= \tfrac{81}{4}\left[\left(\tfrac{\pi}{2}+\tfrac{\sqrt{3}}{2}-\tfrac{\sqrt{3}}{16}\right)-\left(\tfrac{3\pi}{8}+1+0\right)\right] = \tfrac{81}{4}\left(\tfrac{\pi}{8}+\tfrac{7}{16}\sqrt{3}-1\right)$$

17. Let $u = x^2-7$, so $du = 2x\,dx$. Then $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx = \frac{1}{2}\int\frac{1}{\sqrt{u}}\,du = \tfrac{1}{2}\cdot 2\sqrt{u}+C = \sqrt{x^2-7}+C.$

18. Let $ax = b\sec\theta$, so $(ax)^2 = b^2\sec^2\theta \;\Rightarrow$

$(ax)^2 - b^2 = b^2\sec^2\theta - b^2 = b^2(\sec^2\theta - 1) = b^2\tan^2\theta$.

So $\sqrt{(ax)^2 - b^2} = b\tan\theta$, $dx = \dfrac{b}{a}\sec\theta\,\tan\theta\,d\theta$, and

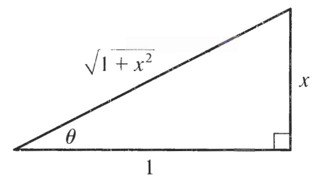


$$\int \frac{dx}{[(ax)^2 - b^2]^{3/2}} = \int \frac{\frac{b}{a}\sec\theta\,\tan\theta}{b^3\tan^3\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{ab^2}\int \csc\theta\,\cot\theta\,d\theta$$

$$= -\frac{1}{ab^2}\csc\theta + C = -\frac{1}{ab^2}\frac{ax}{\sqrt{(ax)^2 - b^2}} + C = -\frac{x}{b^2\sqrt{(ax)^2 - b^2}} + C$$

19. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$

and $\sqrt{1 + x^2} = \sec\theta$, so



$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int \frac{\sec\theta}{\tan\theta}(1 + \tan^2\theta)\,d\theta$$

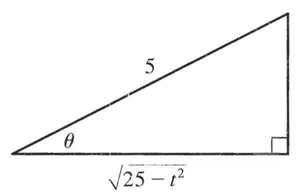
$$= \int(\csc\theta + \sec\theta\,\tan\theta)\,d\theta$$

$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C \qquad \text{[by Exercise 7.2.39]}$$

$$= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2} - 1}{x}\right| + \sqrt{1+x^2} + C$$

20. Let $t = 5\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dt = 5\cos\theta\,d\theta$

and $\sqrt{25 - t^2} = 5\cos\theta$, so

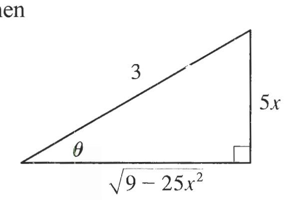


$$\int \frac{t}{\sqrt{25 - t^2}}\,dt = \int \frac{5\sin\theta}{5\cos\theta}\,5\cos\theta\,d\theta = 5\int \sin\theta\,d\theta$$

$$= -5\cos\theta + C = -5\cdot\frac{\sqrt{25 - t^2}}{5} + C = -\sqrt{25 - t^2} + C$$

Or: Let $u = 25 - t^2$, so $du = -2t\,dt$.

21. Let $x = \frac{3}{5}\sin\theta$, so $dx = \frac{3}{5}\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 0.6 \;\Rightarrow\; \theta = \frac{\pi}{2}$. Then



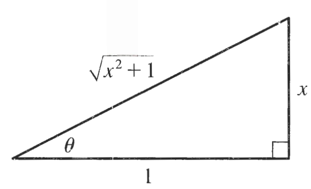
$$\int_0^{0.6} \frac{x^2}{\sqrt{9 - 25x^2}}\,dx = \int_0^{\pi/2} \frac{\left(\frac{3}{5}\right)^2\sin^2\theta}{3\cos\theta}\left(\frac{3}{5}\cos\theta\,d\theta\right) = \frac{9}{125}\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= \tfrac{9}{125}\int_0^{\pi/2}\tfrac{1}{2}(1 - \cos 2\theta)\,d\theta = \tfrac{9}{250}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \tfrac{9}{250}\left[\left(\tfrac{\pi}{2} - 0\right) - 0\right] = \tfrac{9}{500}\pi$$

22. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$,

$\sqrt{x^2 + 1} = \sec\theta$ and $x = 0 \;\Rightarrow\; \theta = 0$, $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$, so

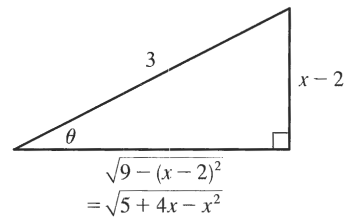


$$\int_0^1 \sqrt{x^2 + 1}\,dx = \int_0^{\pi/4}\sec\theta\,\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^3\theta\,d\theta$$

$$= \tfrac{1}{2}\Big[\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \qquad \text{[by Example 7.2.8]}$$

$$= \tfrac{1}{2}\left[\sqrt{2}\cdot 1 + \ln\left(1 + \sqrt{2}\right) - 0 - \ln(1 + 0)\right] = \tfrac{1}{2}\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$$

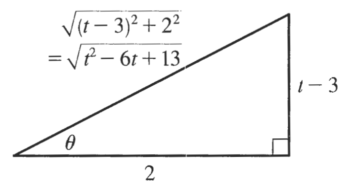
23. $5 + 4x - x^2 = -(x^2 - 4x + 4) + 9 = -(x-2)^2 + 9$. Let $x - 2 = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so $dx = 3\cos\theta\, d\theta$. Then



$$\begin{aligned}\int \sqrt{5+4x-x^2}\,dx &= \int \sqrt{9-(x-2)^2}\,dx = \int \sqrt{9-9\sin^2\theta}\,3\cos\theta\,d\theta \\ &= \int \sqrt{9\cos^2\theta}\,3\cos\theta\,d\theta = \int 9\cos^2\theta\,d\theta \\ &= \tfrac{9}{2}\int(1+\cos 2\theta)\,d\theta = \tfrac{9}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C \\ &= \tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta + C = \tfrac{9}{2}\theta + \tfrac{9}{4}(2\sin\theta\,\cos\theta) + C \\ &= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{5+4x-x^2}}{3} + C \\ &= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C\end{aligned}$$

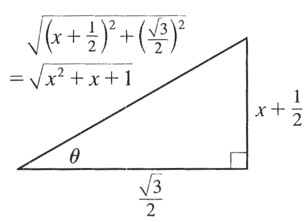
24. $t^2 - 6t + 13 = (t^2 - 6t + 9) + 4 = (t-3)^2 + 2^2$.

Let $t - 3 = 2\tan\theta$, so $dt = 2\sec^2\theta\, d\theta$. Then



$$\begin{aligned}\int \frac{dt}{\sqrt{t^2-6t+13}} &= \int \frac{1}{\sqrt{(2\tan\theta)^2+2^2}}\,2\sec^2\theta\,d\theta \\ &= \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1 \qquad \text{[by Formula 7.2.1]} \\ &= \ln\left|\frac{\sqrt{t^2-6t+13}}{2} + \frac{t-3}{2}\right| + C_1 \\ &= \ln\left|\sqrt{t^2-6t+13} + t - 3\right| + C \qquad \text{where } C = C_1 - \ln 2\end{aligned}$$

25. $x^2 + x + 1 = \left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$. Let $x + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$, so $dx = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$ and $\sqrt{x^2+x+1} = \frac{\sqrt{3}}{2}\sec\theta$. Then

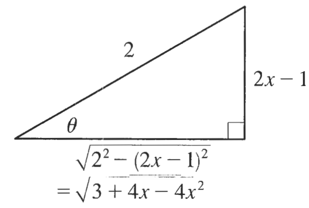


$$\begin{aligned}\int \frac{x}{\sqrt{x^2+x+1}}\,dx &= \int \frac{\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}}{\frac{\sqrt{3}}{2}\sec\theta}\,\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta \\ &= \int\left(\tfrac{\sqrt{3}}{2}\tan\theta - \tfrac{1}{2}\right)\sec\theta\,d\theta = \int \tfrac{\sqrt{3}}{2}\tan\theta\,\sec\theta\,d\theta - \int \tfrac{1}{2}\sec\theta\,d\theta \\ &= \tfrac{\sqrt{3}}{2}\sec\theta - \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C_1 \\ &= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}}\sqrt{x^2+x+1} + \tfrac{2}{\sqrt{3}}\left(x+\tfrac{1}{2}\right)\right| + C_1 \\ &= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}}\left[\sqrt{x^2+x+1} + \left(x+\tfrac{1}{2}\right)\right]\right| + C_1 \\ &= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\tfrac{2}{\sqrt{3}} - \tfrac{1}{2}\ln\left(\sqrt{x^2+x+1} + x + \tfrac{1}{2}\right) + C_1 \\ &= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\left(\sqrt{x^2+x+1} + x + \tfrac{1}{2}\right) + C, \quad \text{where } C = C_1 - \tfrac{1}{2}\ln\tfrac{2}{\sqrt{3}}\end{aligned}$$

26. $3 + 4x - 4x^2 = -(4x^2 - 4x + 1) + 4 = 2^2 - (2x-1)^2$.

Let $2x - 1 = 2\sin\theta$, so $2\,dx = 2\cos\theta\,d\theta$ and $\sqrt{3+4x-4x^2} = 2\cos\theta$.

Then



$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx = \int \frac{\left[\frac{1}{2}(1+2\sin\theta)\right]^2}{(2\cos\theta)^3}\cos\theta\,d\theta$$

$$= \frac{1}{32}\int \frac{1+4\sin\theta+4\sin^2\theta}{\cos^2\theta}\,d\theta = \frac{1}{32}\int(\sec^2\theta + 4\tan\theta\,\sec\theta + 4\tan^2\theta)\,d\theta$$

$$= \tfrac{1}{32}\int[\sec^2\theta + 4\tan\theta\,\sec\theta + 4(\sec^2\theta - 1)]\,d\theta$$

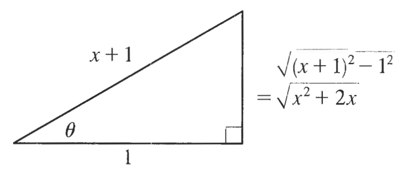
$$= \tfrac{1}{32}\int(5\sec^2\theta + 4\tan\theta\,\sec\theta - 4)\,d\theta = \tfrac{1}{32}(5\tan\theta + 4\sec\theta - 4\theta) + C$$

$$= \frac{1}{32}\left[5\cdot\frac{2x-1}{\sqrt{3+4x-4x^2}} + 4\cdot\frac{2}{\sqrt{3+4x-4x^2}} - 4\cdot\sin^{-1}\left(\frac{2x-1}{2}\right)\right] + C$$

$$= \frac{10x+3}{32\sqrt{3+4x-4x^2}} - \frac{1}{8}\sin^{-1}\left(\frac{2x-1}{2}\right) + C$$

27. $x^2 + 2x = (x^2 + 2x + 1) - 1 = (x+1)^2 - 1$. Let $x + 1 = 1\sec\theta$,

so $dx = \sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2+2x} = \tan\theta$. Then



$$\int \sqrt{x^2+2x}\,dx = \int \tan\theta\,(\sec\theta\,\tan\theta\,d\theta) = \int \tan^2\theta\,\sec\theta\,d\theta$$

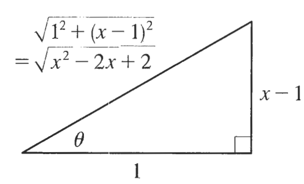
$$= \int(\sec^2\theta - 1)\,\sec\theta\,d\theta = \int\sec^3\theta\,d\theta - \int\sec\theta\,d\theta$$

$$= \tfrac{1}{2}\sec\theta\,\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| - \ln|\sec\theta + \tan\theta| + C$$

$$= \tfrac{1}{2}\sec\theta\,\tan\theta - \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C = \tfrac{1}{2}(x+1)\sqrt{x^2+2x} - \tfrac{1}{2}\ln\left|x + 1 + \sqrt{x^2+2x}\right| + C$$

28. $x^2 - 2x + 2 = (x^2 - 2x + 1) + 1 = (x-1)^2 + 1$. Let $x - 1 = 1\tan\theta$,

so $dx = \sec^2\theta\,d\theta$ and $\sqrt{x^2-2x+2} = \sec\theta$. Then



$$\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx = \int \frac{(\tan\theta+1)^2+1}{\sec^4\theta}\sec^2\theta\,d\theta$$

$$= \int \frac{\tan^2\theta + 2\tan\theta + 2}{\sec^2\theta}\,d\theta$$

$$= \int(\sin^2\theta + 2\sin\theta\,\cos\theta + 2\cos^2\theta)\,d\theta = \int(1 + 2\sin\theta\,\cos\theta + \cos^2\theta)\,d\theta$$

$$= \int\left[1 + 2\sin\theta\,\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]\,d\theta = \int\left(\tfrac{3}{2} + 2\sin\theta\,\cos\theta + \tfrac{1}{2}\cos 2\theta\right)\,d\theta$$

$$= \tfrac{3}{2}\theta + \sin^2\theta + \tfrac{1}{4}\sin 2\theta + C = \tfrac{3}{2}\theta + \sin^2\theta + \tfrac{1}{2}\sin\theta\,\cos\theta + C$$

$$= \frac{3}{2}\tan^{-1}\left(\frac{x-1}{1}\right) + \frac{(x-1)^2}{x^2-2x+2} + \frac{1}{2}\frac{x-1}{\sqrt{x^2-2x+2}}\frac{1}{\sqrt{x^2-2x+2}} + C$$

$$= \frac{3}{2}\tan^{-1}(x-1) + \frac{2(x^2-2x+1)+x-1}{2(x^2-2x+2)} + C = \frac{3}{2}\tan^{-1}(x-1) + \frac{2x^2-3x+1}{2(x^2-2x+2)} + C$$

[continued]

We can write the answer as

$$\frac{3}{2}\tan^{-1}(x-1)+\frac{(2x^2-4x+4)+x-3}{2(x^2-2x+2)}+C=\frac{3}{2}\tan^{-1}(x-1)+1+\frac{x-3}{2(x^2-2x+2)}+C$$
$$=\frac{3}{2}\tan^{-1}(x-1)+\frac{x-3}{2(x^2-2x+2)}+C_1,\ \text{where } C_1=1+C$$

29. Let $u=x^2$, $du=2x\,dx$. Then

$$\int x\sqrt{1-x^4}\,dx=\int\sqrt{1-u^2}\left(\tfrac{1}{2}\,du\right)=\tfrac{1}{2}\int\cos\theta\cdot\cos\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } u=\sin\theta,\ du=\cos\theta\,d\theta,\\ \text{and } \sqrt{1-u^2}=\cos\theta\end{array}\right]$$
$$=\tfrac{1}{2}\int\tfrac{1}{2}(1+\cos 2\theta)\,d\theta=\tfrac{1}{4}\theta+\tfrac{1}{8}\sin 2\theta+C=\tfrac{1}{4}\theta+\tfrac{1}{4}\sin\theta\,\cos\theta+C$$
$$=\tfrac{1}{4}\sin^{-1}u+\tfrac{1}{4}u\sqrt{1-u^2}+C=\tfrac{1}{4}\sin^{-1}(x^2)+\tfrac{1}{4}x^2\sqrt{1-x^4}+C$$

30. Let $u=\sin t$, $du=\cos t\,dt$. Then

$$\int_0^{\pi/2}\frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt=\int_0^1\frac{1}{\sqrt{1+u^2}}\,du=\int_0^{\pi/4}\frac{1}{\sec\theta}\sec^2\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } u=\tan\theta,\ du=\sec^2\theta\,d\theta,\\ \text{and } \sqrt{1+u^2}=\sec\theta\end{array}\right]$$
$$=\int_0^{\pi/4}\sec\theta\,d\theta=\Big[\ln|\sec\theta+\tan\theta|\Big]_0^{\pi/4} \qquad \text{[by (1) in Section 7.2]}$$
$$=\ln\left(\sqrt{2}+1\right)-\ln(1+0)=\ln\left(\sqrt{2}+1\right)$$

31. (a) Let $x=a\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $\sqrt{x^2+a^2}=a\sec\theta$ and

$$\int\frac{dx}{\sqrt{x^2+a^2}}=\int\frac{a\sec^2\theta\,d\theta}{a\sec\theta}=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C_1=\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|+C_1$$
$$=\ln\left(x+\sqrt{x^2+a^2}\right)+C \quad \text{where } C=C_1-\ln|a|$$

(b) Let $x=a\sinh t$, so that $dx=a\cosh t\,dt$ and $\sqrt{x^2+a^2}=a\cosh t$. Then

$$\int\frac{dx}{\sqrt{x^2+a^2}}=\int\frac{a\cosh t\,dt}{a\cosh t}=t+C=\sinh^{-1}\frac{x}{a}+C.$$

32. (a) Let $x=a\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then

$$I=\int\frac{x^2}{(x^2+a^2)^{3/2}}\,dx=\int\frac{a^2\tan^2\theta}{a^3\sec^3\theta}\,a\sec^2\theta\,d\theta=\int\frac{\tan^2\theta}{\sec\theta}\,d\theta=\int\frac{\sec^2\theta-1}{\sec\theta}\,d\theta$$
$$=\int(\sec\theta-\cos\theta)\,d\theta=\ln|\sec\theta+\tan\theta|-\sin\theta+C$$
$$=\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|-\frac{x}{\sqrt{x^2+a^2}}+C=\ln\left(x+\sqrt{x^2+a^2}\right)-\frac{x}{\sqrt{x^2+a^2}}+C_1$$

(b) Let $x=a\sinh t$. Then

$$I=\int\frac{a^2\sinh^2 t}{a^3\cosh^3 t}\,a\cosh t\,dt=\int\tanh^2 t\,dt=\int(1-\operatorname{sech}^2 t)\,dt=t-\tanh t+C$$
$$=\sinh^{-1}\frac{x}{a}-\frac{x}{\sqrt{a^2+x^2}}+C$$

33. The average value of $f(x) = \sqrt{x^2-1}/x$ on the interval $[1, 7]$ is

$$\frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{6}\int_0^\alpha \frac{\tan\theta}{\sec\theta}\cdot\sec\theta\tan\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } x=\sec\theta,\ dx=\sec\theta\tan\theta\,d\theta,\\ \sqrt{x^2-1}=\tan\theta, \text{ and } \alpha=\sec^{-1}7\end{array}\right]$$

$$= \tfrac{1}{6}\int_0^\alpha \tan^2\theta\,d\theta = \tfrac{1}{6}\int_0^\alpha (\sec^2\theta-1)\,d\theta = \tfrac{1}{6}\big[\tan\theta-\theta\big]_0^\alpha$$

$$= \tfrac{1}{6}(\tan\alpha-\alpha) = \tfrac{1}{6}\big(\sqrt{48}-\sec^{-1}7\big)$$

34. $9x^2 - 4y^2 = 36 \ \Rightarrow\ y = \pm\frac{3}{2}\sqrt{x^2-4} \ \Rightarrow$

$$\text{area} = 2\int_2^3 \tfrac{3}{2}\sqrt{x^2-4}\,dx = 3\int_2^3 \sqrt{x^2-4}\,dx$$

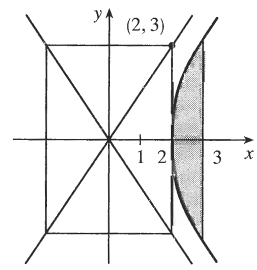
$$= 3\int_0^\alpha 2\tan\theta\,2\sec\theta\,\tan\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } x=2\sec\theta,\\ dx=2\sec\theta\,\tan\theta\,d\theta,\\ \alpha=\sec^{-1}\left(\frac{3}{2}\right)\end{array}\right]$$

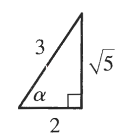
$$= 12\int_0^\alpha \left(\sec^2\theta-1\right)\sec\theta\,d\theta = 12\int_0^\alpha \left(\sec^3\theta-\sec\theta\right)d\theta$$

$$= 12\left[\tfrac{1}{2}(\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|) - \ln|\sec\theta+\tan\theta|\right]_0^\alpha$$

$$= 6\Big[\sec\theta\,\tan\theta - \ln|\sec\theta+\tan\theta|\Big]_0^\alpha$$

$$= 6\left[\tfrac{3\sqrt{5}}{4} - \ln\left(\tfrac{3}{2}+\tfrac{\sqrt{5}}{2}\right)\right] = \tfrac{9\sqrt{5}}{2} - 6\ln\left(\tfrac{3+\sqrt{5}}{2}\right)$$





35. Area of $\triangle POQ = \frac{1}{2}(r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2\sin\theta\,\cos\theta$. Area of region $PQR = \int_{r\cos\theta}^{r}\sqrt{r^2-x^2}\,dx$.

Let $x = r\cos u \ \Rightarrow\ dx = -r\sin u\,du$ for $\theta \le u \le \frac{\pi}{2}$. Then we obtain

$$\int\sqrt{r^2-x^2}\,dx = \int r\sin u\,(-r\sin u)\,du = -r^2\int\sin^2 u\,du = -\tfrac{1}{2}r^2(u - \sin u\,\cos u) + C$$

$$= -\tfrac{1}{2}r^2\cos^{-1}(x/r) + \tfrac{1}{2}x\sqrt{r^2-x^2} + C$$

so

$$\text{area of region } PQR = \tfrac{1}{2}\left[-r^2\cos^{-1}(x/r) + x\sqrt{r^2-x^2}\,\right]_{r\cos\theta}^{r}$$

$$= \tfrac{1}{2}\left[0 - (-r^2\theta + r\cos\theta\, r\sin\theta)\right] = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta\,\cos\theta$$

and thus, (area of sector POR) = (area of $\triangle POQ$) + (area of region PQR) $= \frac{1}{2}r^2\theta$.

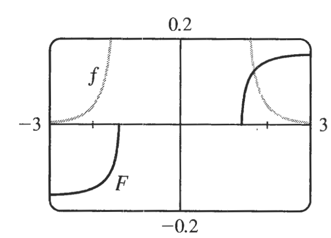
36. Let $x = \sqrt{2}\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, so $dx = \sqrt{2}\,\sec\theta\tan\theta\,d\theta$. Then

$$\int\frac{dx}{x^4\sqrt{x^2-2}} = \int\frac{\sqrt{2}\sec\theta\,\tan\theta\,d\theta}{4\sec^4\theta\,\sqrt{2}\,\tan\theta}$$

$$= \tfrac{1}{4}\int\cos^3\theta\,d\theta = \tfrac{1}{4}\int\left(1-\sin^2\theta\right)\cos\theta\,d\theta$$

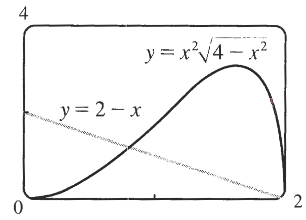
$$= \tfrac{1}{4}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right] + C \qquad [\text{substitute } u = \sin\theta]$$

$$= \frac{1}{4}\left[\frac{\sqrt{x^2-2}}{x} - \frac{\left(x^2-2\right)^{3/2}}{3x^3}\right] + C$$



From the graph, it appears that our answer is reasonable. [Notice that $f(x)$ is large when F increases rapidly and small when F levels out.]

37. From the graph, it appears that the curve $y = x^2\sqrt{4-x^2}$ and the line $y = 2 - x$ intersect at about $x = a \approx 0.81$ and $x = 2$, with $x^2\sqrt{4-x^2} > 2 - x$ on $(a, 2)$. So the area bounded by the curve and the line is $A \approx \int_a^2 \left[x^2\sqrt{4-x^2} - (2-x)\right]dx = \int_a^2 x^2\sqrt{4-x^2}\,dx - \left[2x - \frac{1}{2}x^2\right]_a^2$.



To evaluate the integral, we put $x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then

$dx = 2\cos\theta\,d\theta$, $x = 2 \Rightarrow \theta = \sin^{-1}1 = \frac{\pi}{2}$, and $x = a \Rightarrow \theta = \alpha = \sin^{-1}(a/2) \approx 0.416$. So

$$\int_a^2 x^2\sqrt{4-x^2}\,dx \approx \int_\alpha^{\pi/2} 4\sin^2\theta\,(2\cos\theta)(2\cos\theta\,d\theta) = 4\int_\alpha^{\pi/2}\sin^2 2\theta\,d\theta = 4\int_\alpha^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= 2\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_\alpha^{\pi/2} = 2\left[\left(\tfrac{\pi}{2} - 0\right) - \left(\alpha - \tfrac{1}{4}(0.996)\right)\right] \approx 2.81$$

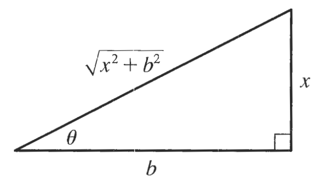
Thus, $A \approx 2.81 - \left[\left(2\cdot 2 - \frac{1}{2}\cdot 2^2\right) - \left(2a - \frac{1}{2}a^2\right)\right] \approx 2.10$.

38. Let $x = b\tan\theta$, so that $dx = b\sec^2\theta\,d\theta$ and $\sqrt{x^2+b^2} = b\sec\theta$.

$$E(P) = \int_{-a}^{L-a}\frac{\lambda b}{4\pi\varepsilon_0(x^2+b^2)^{3/2}}\,dx = \frac{\lambda b}{4\pi\varepsilon_0}\int_{\theta_1}^{\theta_2}\frac{1}{(b\sec\theta)^3}\,b\sec^2\theta\,d\theta$$

$$= \frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\frac{1}{\sec\theta}\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 b}\Big[\sin\theta\Big]_{\theta_1}^{\theta_2}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 b}\left[\frac{x}{\sqrt{x^2+b^2}}\right]_{-a}^{L-a} = \frac{\lambda}{4\pi\varepsilon_0 b}\left(\frac{L-a}{\sqrt{(L-a)^2+b^2}} + \frac{a}{\sqrt{a^2+b^2}}\right)$$



39. (a) Let $t = a\sin\theta$, $dt = a\cos\theta\,d\theta$, $t = 0 \Rightarrow \theta = 0$ and $t = x \Rightarrow \theta = \sin^{-1}(x/a)$. Then

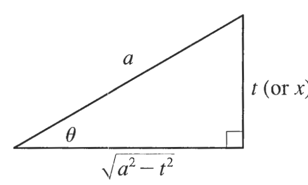
$$\int_0^x \sqrt{a^2-t^2}\,dt = \int_0^{\sin^{-1}(x/a)} a\cos\theta\,(a\cos\theta\,d\theta)$$

$$= a^2\int_0^{\sin^{-1}(x/a)}\cos^2\theta\,d\theta = \frac{a^2}{2}\int_0^{\sin^{-1}(x/a)}(1+\cos 2\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\sin^{-1}(x/a)} = \frac{a^2}{2}\Big[\theta + \sin\theta\,\cos\theta\Big]_0^{\sin^{-1}(x/a)}$$

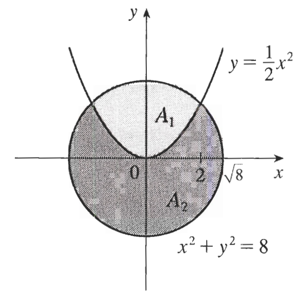
$$= \frac{a^2}{2}\left[\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right) - 0\right]$$

$$= \tfrac{1}{2}a^2\sin^{-1}(x/a) + \tfrac{1}{2}x\sqrt{a^2-x^2}$$



(b) The integral $\int_0^x \sqrt{a^2-t^2}\,dt$ represents the area under the curve $y = \sqrt{a^2-t^2}$ between the vertical lines $t = 0$ and $t = x$. The figure shows that this area consists of a triangular region and a sector of the circle $t^2 + y^2 = a^2$. The triangular region has base x and height $\sqrt{a^2-x^2}$, so its area is $\frac{1}{2}x\sqrt{a^2-x^2}$. The sector has area $\frac{1}{2}a^2\theta = \frac{1}{2}a^2\sin^{-1}(x/a)$.

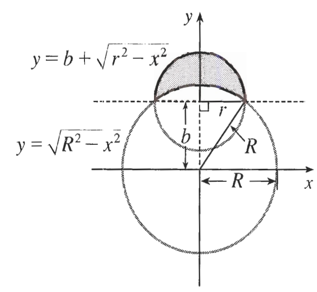
40. The curves intersect when $x^2 + \left(\frac{1}{2}x^2\right)^2 = 8 \Leftrightarrow x^2 + \frac{1}{4}x^4 = 8 \Leftrightarrow x^4 + 4x^2 - 32 = 0 \Leftrightarrow$ $(x^2+8)(x^2-4) = 0 \Leftrightarrow x = \pm 2$. The area inside the circle and above the parabola is given by



$$A_1 = \int_{-2}^{2} \left(\sqrt{8-x^2} - \tfrac{1}{2}x^2\right) dx = 2\int_0^2 \sqrt{8-x^2}\,dx - 2\int_0^2 \tfrac{1}{2}x^2\,dx$$
$$= 2\left[\tfrac{1}{2}(8)\sin^{-1}\left(\tfrac{2}{\sqrt{8}}\right) + \tfrac{1}{2}(2)\sqrt{8-2^2} - \tfrac{1}{2}\left[\tfrac{1}{3}x^3\right]_0^2\right] \qquad \text{[by Exercise 39]}$$
$$= 8\sin^{-1}\left(\tfrac{1}{\sqrt{2}}\right) + 2\sqrt{4} - \tfrac{8}{3} = 8\left(\tfrac{\pi}{4}\right) + 4 - \tfrac{8}{3} = 2\pi + \tfrac{4}{3}$$

Since the area of the disk is $\pi\left(\sqrt{8}\right)^2 = 8\pi$, the area inside the circle and below the parabola ia $A_2 = 8\pi - \left(2\pi + \frac{4}{3}\right) = 6\pi - \frac{4}{3}$.

41. Let the equation of the large circle be $x^2 + y^2 = R^2$. Then the equation of the small circle is $x^2 + (y-b)^2 = r^2$, where $b = \sqrt{R^2 - r^2}$ is the distance between the centers of the circles. The desired area is



$$A = \int_{-r}^{r} \left[\left(b + \sqrt{r^2-x^2}\right) - \sqrt{R^2-x^2}\right] dx$$
$$= 2\int_0^r \left(b + \sqrt{r^2-x^2} - \sqrt{R^2-x^2}\right) dx$$
$$= 2\int_0^r b\,dx + 2\int_0^r \sqrt{r^2-x^2}\,dx - 2\int_0^r \sqrt{R^2-x^2}\,dx$$

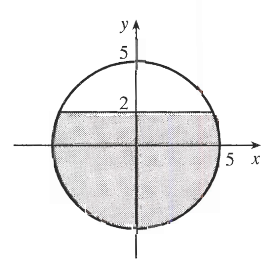
The first integral is just $2br = 2r\sqrt{R^2-r^2}$. The second integral represents the area of a quarter-circle of radius r, so its value is $\frac{1}{4}\pi r^2$. To evaluate the other integral, note that

$$\int \sqrt{a^2-x^2}\,dx = \int a^2\cos^2\theta\,d\theta \quad [x = a\sin\theta,\, dx = a\cos\theta\,d\theta] \quad = \left(\tfrac{1}{2}a^2\right)\int(1+\cos 2\theta)\,d\theta$$
$$= \tfrac{1}{2}a^2\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{2}a^2(\theta + \sin\theta\,\cos\theta) + C$$
$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a} + C = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C$$

Thus, the desired area is

$$A = 2r\sqrt{R^2-r^2} + 2\left(\tfrac{1}{4}\pi r^2\right) - \left[R^2\arcsin(x/R) + x\sqrt{R^2-x^2}\right]_0^r$$
$$= 2r\sqrt{R^2-r^2} + \tfrac{1}{2}\pi r^2 - \left[R^2\arcsin(r/R) + r\sqrt{R^2-r^2}\right] = r\sqrt{R^2-r^2} + \tfrac{\pi}{2}r^2 - R^2\arcsin(r/R)$$

42. Note that the circular cross-sections of the tank are the same everywhere, so the percentage of the total capacity that is being used is equal to the percentage of any cross-section that is under water. The underwater area is



$$A = 2\int_{-5}^{2} \sqrt{25-y^2}\,dy$$
$$= \left[25\arcsin(y/5) + y\sqrt{25-y^2}\right]_{-5}^{2} \qquad \text{[substitute } y = 5\sin\theta\text{]}$$
$$= 25\arcsin\tfrac{2}{5} + 2\sqrt{21} + \tfrac{25}{2}\pi \approx 58.72 \text{ ft}^2$$

so the fraction of the total capacity in use is $\dfrac{A}{\pi(5)^2} \approx \dfrac{58.72}{25\pi} \approx 0.748$ or 74.8%.

43. We use cylindrical shells and assume that $R > r$. $x^2 = r^2 - (y-R)^2 \Rightarrow x = \pm\sqrt{r^2 - (y-R)^2}$,

so $g(y) = 2\sqrt{r^2 - (y-R)^2}$ and

$$V = \int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2 - (y-R)^2}\,dy = \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2 - u^2}\,du \qquad [\text{where } u = y - R]$$

$$= 4\pi \int_{-r}^{r} u\sqrt{r^2 - u^2}\,du + 4\pi R \int_{-r}^{r} \sqrt{r^2 - u^2}\,du \qquad \left[\begin{array}{c}\text{where } u = r\sin\theta,\ du = r\cos\theta\,d\theta \\ \text{in the second integral}\end{array}\right]$$

$$= 4\pi\left[-\tfrac{1}{3}(r^2 - u^2)^{3/2}\right]_{-r}^{r} + 4\pi R \int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\,d\theta = -\tfrac{4\pi}{3}(0-0) + 4\pi R r^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta$$

$$= 2\pi R r^2 \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\,d\theta = 2\pi R r^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2\pi^2 R r^2$$

Another method: Use washers instead of shells, so $V = 8\pi R \int_0^r \sqrt{r^2 - y^2}\,dy$ as in Exercise 6.2.63(a), but evaluate the integral using $y = r\sin\theta$.