1. (a) $\dfrac{2x}{(x+3)(3x+1)} = \dfrac{A}{x+3} + \dfrac{B}{3x+1}$

(b) $\dfrac{1}{x^3+2x^2+x} = \dfrac{1}{x(x^2+2x+1)} = \dfrac{1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

2. (a) $\dfrac{x}{x^2+x-2} = \dfrac{x}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1}$

(b) $\dfrac{x^2}{x^2+x+2} = \dfrac{(x^2+x+2)-(x+2)}{x^2+x+2} = 1 - \dfrac{x+2}{x^2+x+2}$

Notice that x^2+x+2 can't be factored because its discriminant is $b^2-4ac=-7<0$.

3. (a) $\dfrac{x^4+1}{x^5+4x^3} = \dfrac{x^4+1}{x^3(x^2+4)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+4}$

(b) $\dfrac{1}{(x^2-9)^2} = \dfrac{1}{[(x+3)(x-3)]^2} = \dfrac{1}{(x+3)^2(x-3)^2} = \dfrac{A}{x+3} + \dfrac{B}{(x+3)^2} + \dfrac{C}{x-3} + \dfrac{D}{(x-3)^2}$

4. (a) $\dfrac{x^3}{x^2+4x+3} = x-4+\dfrac{13x+12}{x^2+4x+3} = x-4+\dfrac{13x+12}{(x+1)(x+3)} = x-4+\dfrac{A}{x+1} + \dfrac{B}{x+3}$

(b) $\dfrac{2x+1}{(x+1)^3(x^2+4)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3} + \dfrac{Dx+E}{x^2+4} + \dfrac{Fx+G}{(x^2+4)^2}$

5. (a) $\dfrac{x^4}{x^4-1} = \dfrac{(x^4-1)+1}{x^4-1} = 1 + \dfrac{1}{x^4-1}$ [or use long division] $= 1 + \dfrac{1}{(x^2-1)(x^2+1)}$

$= 1 + \dfrac{1}{(x-1)(x+1)(x^2+1)} = 1 + \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1}$

(b) $\dfrac{t^4+t^2+1}{(t^2+1)(t^2+4)^2} = \dfrac{At+B}{t^2+1} + \dfrac{Ct+D}{t^2+4} + \dfrac{Et+F}{(t^2+4)^2}$

6. (a) $\dfrac{x^4}{(x^3+x)(x^2-x+3)} = \dfrac{x^4}{x(x^2+1)(x^2-x+3)} = \dfrac{x^3}{(x^2+1)(x^2-x+3)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2-x+3}$

(b) $\dfrac{1}{x^6-x^3} = \dfrac{1}{x^3(x^3-1)} = \dfrac{1}{x^3(x-1)(x^2+x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x-1} + \dfrac{Ex+F}{x^2+x+1}$

7. $\displaystyle\int \frac{x}{x-6}\,dx = \int \frac{(x-6)+6}{x-6}dx = \int \left(1+\frac{6}{x-6}\right)dx = x + 6\ln|x-6| + C$

8. $\displaystyle\int \frac{r^2}{r+4}\,dr = \int \left(\frac{r^2-16}{r+4} + \frac{16}{r+4}\right)dr = \int \left(r-4+\frac{16}{r+4}\right)dr$ [or use long division]

$= \frac{1}{2}r^2 - 4r + 16\ln|r+4| + C$

9. $\dfrac{x-9}{(x+5)(x-2)} = \dfrac{A}{x+5} + \dfrac{B}{x-2}$. Multiply both sides by $(x+5)(x-2)$ to get $x-9 = A(x-2)+B(x+5)\,(*)$, or equivalently, $x-9=(A+B)x-2A+5B$. Equating coefficients of x on each side of the equation gives us $1=A+B$ **(1)** and equating constants gives us $-9=-2A+5B$ **(2)**. Adding two times **(1)** to **(2)** gives us $-7=7B \;\Leftrightarrow\; B=-1$ and hence, $A=2$. [Alternatively, to find the coefficients A and B, we may use substitution as follows: substitute 2 for x in $(*)$ to get $-7=7B \;\Leftrightarrow\; B=-1$, then substitute -5 for x in $(*)$ to get $-14=-7A \;\Leftrightarrow\; A=2$.] Thus,

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int \left(\frac{2}{x+5} + \frac{-1}{x-2}\right)dx = 2\ln|x+5| - \ln|x-2| + C.$$

10. $\dfrac{1}{(t+4)(t-1)} = \dfrac{A}{t+4} + \dfrac{B}{t-1} \;\Rightarrow\; 1 = A(t-1)+B(t+4).$

$t=1 \;\Rightarrow\; 1=5B \;\Rightarrow\; B=\frac{1}{5}$. $t=-4 \;\Rightarrow\; 1=-5A \;\Rightarrow\; A=-\frac{1}{5}$. Thus,

$$\int \frac{1}{(t+4)(t-1)}\,dt = \int \left(\frac{-1/5}{t+4} + \frac{1/5}{t-1}\right)dt = -\tfrac{1}{5}\ln|t+4| + \tfrac{1}{5}\ln|t-1| + C \quad\text{or}\quad \frac{1}{5}\ln\left|\frac{t-1}{t+4}\right| + C$$

11. $\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$. Multiply both sides by $(x+1)(x-1)$ to get $1=A(x-1)+B(x+1)$. Substituting 1 for x gives $1=2B \;\Leftrightarrow\; B=\frac{1}{2}$. Substituting -1 for x gives $1=-2A \;\Leftrightarrow\; A=-\frac{1}{2}$. Thus,

$$\int_2^3 \frac{1}{x^2-1}\,dx = \int_2^3 \left(\frac{-1/2}{x+1} + \frac{1/2}{x-1}\right)dx = \left[-\tfrac{1}{2}\ln|x+1| + \tfrac{1}{2}\ln|x-1|\right]_2^3$$

$$= \left(-\tfrac{1}{2}\ln 4 + \tfrac{1}{2}\ln 2\right) - \left(-\tfrac{1}{2}\ln 3 + \tfrac{1}{2}\ln 1\right) = \tfrac{1}{2}(\ln 2 + \ln 3 - \ln 4) \quad \left[\text{or } \tfrac{1}{2}\ln\tfrac{3}{2}\right]$$

12. $\dfrac{x-1}{x^2+3x+2} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$. Multiply both sides by $(x+1)(x+2)$ to get $x-1=A(x+2)+B(x+1)$. Substituting -2 for x gives $-3=-B \;\Leftrightarrow\; B=3$. Substituting -1 for x gives $-2=A$. Thus,

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1 \left(\frac{-2}{x+1} + \frac{3}{x+2}\right)dx = \left[-2\ln|x+1| + 3\ln|x+2|\right]_0^1$$

$$= (-2\ln 2 + 3\ln 3) - (-2\ln 1 + 3\ln 2) = 3\ln 3 - 5\ln 2 \quad \left[\text{or } \ln\tfrac{27}{32}\right]$$

13. $\displaystyle\int \frac{ax}{x^2-bx}\,dx = \int \frac{ax}{x(x-b)}\,dx = \int \frac{a}{x-b}\,dx = a\ln|x-b| + C$

14. If $a \neq b$, $\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{b-a}\left(\dfrac{1}{x+a} - \dfrac{1}{x+b}\right)$, so if $a \neq b$, then

$$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C = \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C$$

If $a = b$, then $\displaystyle\int \frac{dx}{(x+a)^2} = -\frac{1}{x+a} + C$.

15. $\dfrac{x^3-2x^2-4}{x^3-2x^2} = 1 + \dfrac{-4}{x^2(x-2)}$. Write $\dfrac{-4}{x^2(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2}$. Multiplying both sides by $x^2(x-2)$ gives $-4 = Ax(x-2) + B(x-2) + Cx^2$. Substituting 0 for x gives $-4 = -2B \;\Leftrightarrow\; B = 2$. Substituting 2 for x gives $-4 = 4C \;\Leftrightarrow\; C = -1$. Equating coefficients of x^2, we get $0 = A + C$, so $A = 1$. Thus,

$$\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx = \int_3^4 \left(1 + \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x-2}\right)dx = \left[x + \ln|x| - \frac{2}{x} - \ln|x-2|\right]_3^4$$
$$= \left[\left(4 + \ln 4 - \tfrac{1}{2} - \ln 2\right) - \left(3 + \ln 3 - \tfrac{2}{3} - 0\right)\right] = \tfrac{7}{6} + \ln\tfrac{2}{3}$$

16. $\dfrac{x^3-4x-10}{x^2-x-6} = x + 1 + \dfrac{3x-4}{(x-3)(x+2)}$. Write $\dfrac{3x-4}{(x-3)(x+2)} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$. Then $3x - 4 = A(x+2) + B(x-3)$. Taking $x = 3$ and $x = -2$, we get $5 = 5A \;\Leftrightarrow\; A = 1$ and $-10 = -5B \;\Leftrightarrow\; B = 2$, so

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx = \int_0^1 \left(x + 1 + \frac{1}{x-3} + \frac{2}{x+2}\right)dx = \left[\frac{1}{2}x^2 + x + \ln|x-3| + 2\ln(x+2)\right]_0^1$$
$$= \left(\tfrac{1}{2} + 1 + \ln 2 + 2\ln 3\right) - \left(0 + 0 + \ln 3 + 2\ln 2\right) = \tfrac{3}{2} + \ln 3 - \ln 2 = \tfrac{3}{2} + \ln\tfrac{3}{2}$$

17. $\dfrac{4y^2-7y-12}{y(y+2)(y-3)} = \dfrac{A}{y} + \dfrac{B}{y+2} + \dfrac{C}{y-3} \;\Rightarrow\; 4y^2 - 7y - 12 = A(y+2)(y-3) + By(y-3) + Cy(y+2)$. Setting $y = 0$ gives $-12 = -6A$, so $A = 2$. Setting $y = -2$ gives $18 = 10B$, so $B = \frac{9}{5}$. Setting $y = 3$ gives $3 = 15C$, so $C = \frac{1}{5}$. Now

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy = \int_1^2 \left(\frac{2}{y} + \frac{9/5}{y+2} + \frac{1/5}{y-3}\right)dy = \left[2\ln|y| + \tfrac{9}{5}\ln|y+2| + \tfrac{1}{5}\ln|y-3|\right]_1^2$$
$$= 2\ln 2 + \tfrac{9}{5}\ln 4 + \tfrac{1}{5}\ln 1 - 2\ln 1 - \tfrac{9}{5}\ln 3 - \tfrac{1}{5}\ln 2$$
$$= 2\ln 2 + \tfrac{18}{5}\ln 2 - \tfrac{1}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{27}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{9}{5}(3\ln 2 - \ln 3) = \tfrac{9}{5}\ln\tfrac{8}{3}$$

18. $\dfrac{x^2+2x-1}{x^3-x} = \dfrac{x^2+2x-1}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$. Multiply both sides by $x(x+1)(x-1)$ to get $x^2 + 2x - 1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$. Substituting 0 for x gives $-1 = -A \;\Leftrightarrow\; A = 1$. Substituting -1 for x gives $-2 = 2B \;\Leftrightarrow\; B = -1$. Substituting 1 for x gives $2 = 2C \;\Leftrightarrow\; C = 1$. Thus,

$$\int \frac{x^2+2x-1}{x^3-x}\,dx = \int \left(\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x-1}\right)dx = \ln|x| - \ln|x+1| + \ln|x-1| + C = \ln\left|\frac{x(x-1)}{x+1}\right| + C.$$

19. $\dfrac{1}{(x+5)^2(x-1)} = \dfrac{A}{x+5} + \dfrac{B}{(x+5)^2} + \dfrac{C}{x-1} \quad \Rightarrow \quad 1 = A(x+5)(x-1) + B(x-1) + C(x+5)^2.$

Setting $x=-5$ gives $1=-6B$, so $B=-\frac{1}{6}$. Setting $x=1$ gives $1=36C$, so $C=\frac{1}{36}$. Setting $x=-2$ gives

$1 = A(3)(-3) + B(-3) + C(3^2) = -9A - 3B + 9C = -9A + \frac{1}{2} + \frac{1}{4} = -9A + \frac{3}{4}$, so $9A = -\frac{1}{4}$ and $A = -\frac{1}{36}$. Now

$$\int \frac{1}{(x+5)^2(x-1)}\,dx = \int \left[\frac{-1/36}{x+5} - \frac{1/6}{(x+5)^2} + \frac{1/36}{x-1}\right] dx = -\frac{1}{36}\ln|x+5| + \frac{1}{6(x+5)} + \frac{1}{36}\ln|x-1| + C.$$

20. $\dfrac{x^2-5x+16}{(2x+1)(x-2)^2} = \dfrac{A}{2x+1} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2} \quad \Rightarrow \quad x^2-5x+16 = A(x-2)^2 + B(x-2)(2x+1) + C(2x+1).$

Setting $x=2$ gives $10=5C$, so $C=2$. Setting $x=-\frac{1}{2}$ gives $\frac{75}{4}=\frac{25}{4}A$, so $A=3$. Equating coefficients of x^2, we get

$1 = A + 2B$, so $-2 = 2B$ and $B = -1$. Thus,

$$\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx = \int \left(\frac{3}{2x+1} - \frac{1}{x-2} + \frac{2}{(x-2)^2}\right) dx = \frac{3}{2}\ln|2x+1| - \ln|x-2| - \frac{2}{x-2} + C$$

21.

$$\begin{array}{r|l} & \quad x \\ x^2+4 & x^3 + 0x^2 + 0x + 4 \\ & x^3 \qquad\quad + 4x \\ \hline & \qquad\quad -4x + 4 \end{array}$$

By long division, $\dfrac{x^3+4}{x^2+4} = x + \dfrac{-4x+4}{x^2+4}$. Thus,

$$\int \frac{x^3+4}{x^2+4}\,dx = \int \left(x + \frac{-4x+4}{x^2+4}\right) dx = \int \left(x - \frac{4x}{x^2+4} + \frac{4}{x^2+2^2}\right) dx$$

$$= \frac{1}{2}x^2 - 4\cdot\frac{1}{2}\ln|x^2+4| + 4\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C = \frac{1}{2}x^2 - 2\ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

22. $\dfrac{1}{s^2(s-1)^2} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-1} + \dfrac{D}{(s-1)^2} \quad \Rightarrow \quad 1 = As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2.$

Set $s=0$, giving $B=1$. Then set $s=1$ to get $D=1$. Equate the coefficients of s^3 to get

$0 = A + C$ or $A = -C$, and finally set $s=2$ to get $1 = 2A + 1 - 4A + 4$ or $A = 2$. Now

$$\int \frac{ds}{s^2(s-1)^2} = \int \left[\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2}\right] ds = 2\ln|s| - \frac{1}{s} - 2\ln|s-1| - \frac{1}{s-1} + C.$$

23. $\dfrac{5x^2+3x-2}{x^3+2x^2} = \dfrac{5x^2+3x-2}{x^2\,(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$. Multiply by $x^2(x+2)$ to

get $5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x=-2$ to get $C=3$, and take

$x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5 = A + C \quad \Rightarrow \quad A = 2$. So

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right) dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$$

24. $\dfrac{x^2-x+6}{x^3+3x}=\dfrac{x^2-x+6}{x(x^2+3)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get $x^2-x+6=A(x^2+3)+(Bx+C)x$.

Substituting 0 for x gives $6=3A \Leftrightarrow A=2$. The coefficients of the x^2-terms must be equal, so $1=A+B \Rightarrow$ $B=1-2=-1$. The coefficients of the x-terms must be equal, so $-1=C$. Thus,

$$\begin{aligned}\int\frac{x^2-x+6}{x^3+3x}\,dx&=\int\left(\frac{2}{x}+\frac{-x-1}{x^2+3}\right)dx=\int\left(\frac{2}{x}-\frac{x}{x^2+3}-\frac{1}{x^2+3}\right)dx\\&=2\ln|x|-\frac{1}{2}\ln(x^2+3)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+C\end{aligned}$$

25. $\dfrac{10}{(x-1)(x^2+9)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)(x^2+9)$ to get

$10=A(x^2+9)+(Bx+C)(x-1)$ $(\star)$. Substituting 1 for x gives $10=10A \Leftrightarrow A=1$. Substituting 0 for x gives $10=9A-C \Rightarrow C=9(1)-10=-1$. The coefficients of the x^2-terms in $(\star)$ must be equal, so $0=A+B \Rightarrow$ $B=-1$. Thus,

$$\begin{aligned}\int\frac{10}{(x-1)(x^2+9)}\,dx&=\int\left(\frac{1}{x-1}+\frac{-x-1}{x^2+9}\right)dx=\int\left(\frac{1}{x-1}-\frac{x}{x^2+9}-\frac{1}{x^2+9}\right)dx\\&=\ln|x-1|-\tfrac{1}{2}\ln(x^2+9)-\tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right)+C\end{aligned}$$

In the second term we used the substitution $u=x^2+9$ and in the last term we used Formula 10.

26. $$\begin{aligned}\int\frac{x^2+x+1}{(x^2+1)^2}\,dx&=\int\frac{x^2+1}{(x^2+1)^2}\,dx+\int\frac{x}{(x^2+1)^2}\,dx=\int\frac{1}{x^2+1}\,dx+\frac{1}{2}\int\frac{1}{u^2}\,du\quad[u=x^2+1,\,du=2x\,dx]\\&=\tan^{-1}x+\frac{1}{2}\left(-\frac{1}{u}\right)+C=\tan^{-1}x-\frac{1}{2(x^2+1)}+C\end{aligned}$$

27. $\dfrac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+2}$. Multiply both sides by $(x^2+1)(x^2+2)$ to get

$x^3+x^2+2x+1=(Ax+B)(x^2+2)+(Cx+D)(x^2+1) \Leftrightarrow$

$x^3+x^2+2x+1=(Ax^3+Bx^2+2Ax+2B)+(Cx^3+Dx^2+Cx+D) \Leftrightarrow$

$x^3+x^2+2x+1=(A+C)x^3+(B+D)x^2+(2A+C)x+(2B+D)$. Comparing coefficients gives us the following system of equations:

$$\begin{array}{llll}A+C=1 & \textbf{(1)} & B+D=1 & \textbf{(2)}\\ 2A+C=2 & \textbf{(3)} & 2B+D=1 & \textbf{(4)}\end{array}$$

Subtracting equation **(1)** from equation **(3)** gives us $A=1$, so $C=0$. Subtracting equation **(2)** from equation **(4)** gives us $B=0$, so $D=1$. Thus, $I=\displaystyle\int\frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx=\int\left(\frac{x}{x^2+1}+\frac{1}{x^2+2}\right)dx$. For $\displaystyle\int\frac{x}{x^2+1}\,dx$, let $u=x^2+1$ so $du=2x\,dx$ and then $\displaystyle\int\frac{x}{x^2+1}\,dx=\frac{1}{2}\int\frac{1}{u}\,du=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln(x^2+1)+C$. For $\displaystyle\int\frac{1}{x^2+2}\,dx$, use

Formula 10 with $a = \sqrt{2}$. So $\displaystyle\int \frac{1}{x^2+2}\,dx = \int \frac{1}{x^2+(\sqrt{2})^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$.

Thus, $I = \dfrac{1}{2}\ln(x^2+1) + \dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}} + C$.

28. $\dfrac{x^2-2x-1}{(x-1)^2(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{Cx+D}{x^2+1} \quad\Rightarrow$

$x^2-2x-1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$. Setting $x=1$ gives $B=-1$. Equating the coefficients of x^3 gives $A=-C$. Equating the constant terms gives $-1=-A-1+D$, so $D=A$, and setting $x=2$ gives $-1=5A-5-2A+A$ or $A=1$. We have

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx = \int\left[\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{x-1}{x^2+1}\right]dx = \ln|x-1| + \frac{1}{x-1} - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + C.$$

29. $$\int \frac{x+4}{x^2+2x+5}\,dx = \int \frac{x+1}{x^2+2x+5}\,dx + \int \frac{3}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{(2x+2)\,dx}{x^2+2x+5} + \int \frac{3\,dx}{(x+1)^2+4}$$

$$= \frac{1}{2}\ln|x^2+2x+5| + 3\int \frac{2\,du}{4(u^2+1)} \qquad \left[\begin{array}{l}\text{where } x+1=2u,\\ \text{and } dx = 2\,du\end{array}\right]$$

$$= \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}u + C = \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

30. $\dfrac{3x^2+x+4}{x^4+3x^2+2} = \dfrac{3x^2+x+4}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2}$. Multiply both sides by $(x^2+1)(x^2+2)$ to get

$3x^2+x+4 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1) \quad\Leftrightarrow$

$3x^2+x+4 = (Ax^3+Bx^2+2Ax+2B) + (Cx^3+Dx^2+Cx+D) \quad\Leftrightarrow$

$3x^2+x+4 = (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$. Comparing coefficients gives us the following system of equations:

$$A+C=0 \quad \textbf{(1)} \qquad B+D=3 \quad \textbf{(2)}$$
$$2A+C=1 \quad \textbf{(3)} \qquad 2B+D=4 \quad \textbf{(4)}$$

Subtracting equation **(1)** from equation **(3)** gives us $A=1$, so $C=-1$. Subtracting equation **(2)** from equation **(4)** gives us $B=1$, so $D=2$. Thus,

$$I = \int \frac{3x^2+x+4}{x^4+3x^2+2}\,dx = \int \frac{x+1}{x^2+1}\,dx + \int \frac{-x+2}{x^2+2}\,dx$$

$$= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx - \frac{1}{2}\int \frac{2x}{x^2+2}\,dx + 2\int \frac{1}{x^2+(\sqrt{2})^2}\,dx$$

$$= \tfrac{1}{2}\ln|x^2+1| + \tan^{-1}x - \tfrac{1}{2}\ln|x^2+2| + 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

$$= \tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{2}\ln(x^2+2) + \tan^{-1}x + \sqrt{2}\tan^{-1}(x/\sqrt{2}) + C$$

31. $\dfrac{1}{x^3-1}=\dfrac{1}{(x-1)(x^2+x+1)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+x+1} \quad\Rightarrow\quad 1=A(x^2+x+1)+(Bx+C)(x-1).$

Take $x=1$ to get $A=\frac{1}{3}$. Equating coefficients of x^2 and then comparing the constant terms, we get $0=\frac{1}{3}+B$, $1=\frac{1}{3}-C$, so $B=-\frac{1}{3}$, $C=-\frac{2}{3}$ $\Rightarrow$

$$\begin{aligned}\int\frac{1}{x^3-1}dx&=\int\frac{\frac{1}{3}}{x-1}\,dx+\int\frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}\,dx=\tfrac{1}{3}\ln|x-1|-\frac{1}{3}\int\frac{x+2}{x^2+x+1}dx\\&=\tfrac{1}{3}\ln|x-1|-\frac{1}{3}\int\frac{x+1/2}{x^2+x+1}dx-\frac{1}{3}\int\frac{(3/2)\,dx}{(x+1/2)^2+3/4}\\&=\tfrac{1}{3}\ln|x-1|-\tfrac{1}{6}\ln(x^2+x+1)-\tfrac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)+K\\&=\tfrac{1}{3}\ln|x-1|-\tfrac{1}{6}\ln(x^2+x+1)-\tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x+1)\right)+K\end{aligned}$$

32. $\displaystyle\int_0^1\frac{x}{x^2+4x+13}\,dx=\int_0^1\frac{\frac{1}{2}(2x+4)}{x^2+4x+13}\,dx-2\int_0^1\frac{dx}{(x+2)^2+9}$

$$\begin{aligned}&=\frac{1}{2}\int_{13}^{18}\frac{dy}{y}-2\int_{2/3}^{1}\frac{3\,du}{9u^2+9}\qquad\left[\begin{array}{c}\text{where } y=x^2+4x+13,\ dy=(2x+4)\,dx,\\ x+2=3u,\text{ and } dx=3\,du\end{array}\right]\\&=\tfrac{1}{2}\left[\ln y\right]_{13}^{18}-\tfrac{2}{3}\left[\tan^{-1}u\right]_{2/3}^{1}=\tfrac{1}{2}\ln\tfrac{18}{13}-\tfrac{2}{3}\left(\tfrac{\pi}{4}-\tan^{-1}\left(\tfrac{2}{3}\right)\right)\\&=\tfrac{1}{2}\ln\tfrac{18}{13}-\tfrac{\pi}{6}+\tfrac{2}{3}\tan^{-1}\left(\tfrac{2}{3}\right)\end{aligned}$$

33. Let $u=x^4+4x^2+3$, so that $du=(4x^3+8x)\,dx=4(x^3+2x)\,dx$, $x=0\ \Rightarrow\ u=3$, and $x=1\ \Rightarrow\ u=8$.

Then $\displaystyle\int_0^1\frac{x^3+2x}{x^4+4x^2+3}\,dx=\int_3^8\frac{1}{u}\left(\frac{1}{4}\,du\right)=\frac{1}{4}\left[\ln|u|\right]_3^8=\frac{1}{4}(\ln 8-\ln 3)=\frac{1}{4}\ln\frac{8}{3}.$

34. $\dfrac{x^3}{x^3+1}=\dfrac{(x^3+1)-1}{x^3+1}=1-\dfrac{1}{x^3+1}=1-\left(\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2-x+1}\right)\quad\Rightarrow\quad 1=A(x^2-x+1)+(Bx+C)(x+1).$

Equate the terms of degree 2, 1 and 0 to get $0=A+B$, $0=-A+B+C$, $1=A+C$. Solve the three equations to get $A=\frac{1}{3}$, $B=-\frac{1}{3}$, and $C=\frac{2}{3}$. So

$$\begin{aligned}\int\frac{x^3}{x^3+1}\,dx&=\int\left[1-\frac{\frac{1}{3}}{x+1}+\frac{\frac{1}{3}x-\frac{2}{3}}{x^2-x+1}\right]dx=x-\tfrac{1}{3}\ln|x+1|+\frac{1}{6}\int\frac{2x-1}{x^2-x+1}\,dx-\frac{1}{2}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\\&=x-\tfrac{1}{3}\ln|x+1|+\tfrac{1}{6}\ln(x^2-x+1)-\tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x-1)\right)+K\end{aligned}$$

35. $\dfrac{1}{x(x^2+4)^2}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+4}+\dfrac{Dx+E}{(x^2+4)^2}\quad\Rightarrow\quad 1=A(x^2+4)^2+(Bx+C)x(x^2+4)+(Dx+E)x$. Setting $x=0$ gives $1=16A$, so $A=\frac{1}{16}$. Now compare coefficients.

$$\begin{aligned}1&=\tfrac{1}{16}(x^4+8x^2+16)+(Bx^2+Cx)(x^2+4)+Dx^2+Ex\\1&=\tfrac{1}{16}x^4+\tfrac{1}{2}x^2+1+Bx^4+Cx^3+4Bx^2+4Cx+Dx^2+Ex\\1&=\left(\tfrac{1}{16}+B\right)x^4+Cx^3+\left(\tfrac{1}{2}+4B+D\right)x^2+(4C+E)x+1\end{aligned}$$

So $B + \frac{1}{16} = 0 \Rightarrow B = -\frac{1}{16}$, $C = 0$, $\frac{1}{2} + 4B + D = 0 \Rightarrow D = -\frac{1}{4}$, and $4C + E = 0 \Rightarrow E = 0$. Thus,

$$\int \frac{dx}{x(x^2+4)^2} = \int \left(\frac{\frac{1}{16}}{x} + \frac{-\frac{1}{16}x}{x^2+4} + \frac{-\frac{1}{4}x}{(x^2+4)^2} \right) dx = \frac{1}{16}\ln|x| - \frac{1}{16}\cdot\frac{1}{2}\ln|x^2+4| - \frac{1}{4}\left(-\frac{1}{2}\right)\frac{1}{x^2+4} + C$$
$$= \frac{1}{16}\ln|x| - \frac{1}{32}\ln(x^2+4) + \frac{1}{8(x^2+4)} + C$$

36. Let $u = x^5 + 5x^3 + 5x$, so that $du = (5x^4 + 15x^2 + 5)dx = 5(x^4 + 3x^2 + 1)dx$. Then

$$\int \frac{x^4+3x^2+1}{x^5+5x^3+5x}\,dx = \int \frac{1}{u}\left(\frac{1}{5}\,du\right) = \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln\left|x^5+5x^3+5x\right| + C$$

37. $\dfrac{x^2-3x+7}{(x^2-4x+6)^2} = \dfrac{Ax+B}{x^2-4x+6} + \dfrac{Cx+D}{(x^2-4x+6)^2} \Rightarrow x^2 - 3x + 7 = (Ax+B)(x^2-4x+6) + Cx + D \Rightarrow$

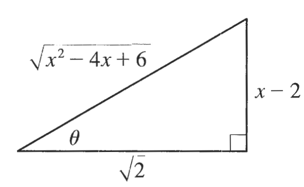
$x^2 - 3x + 7 = Ax^3 + (-4A+B)x^2 + (6A - 4B + C)x + (6B + D)$. So $A = 0$, $-4A + B = 1 \Rightarrow B = 1$, $6A - 4B + C = -3 \Rightarrow C = 1$, $6B + D = 7 \Rightarrow D = 1$. Thus,

$$I = \int \frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx = \int \left(\frac{1}{x^2-4x+6} + \frac{x+1}{(x^2-4x+6)^2} \right) dx$$
$$= \int \frac{1}{(x-2)^2+2}\,dx + \int \frac{x-2}{(x^2-4x+6)^2}\,dx + \int \frac{3}{(x^2-4x+6)^2}\,dx$$
$$= I_1 + I_2 + I_3.$$

$$I_1 = \int \frac{1}{(x-2)^2 + \left(\sqrt{2}\right)^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + C_1$$

$$I_2 = \frac{1}{2}\int \frac{2x-4}{(x^2-4x+6)^2}\,dx = \frac{1}{2}\int \frac{1}{u^2}\,du = \frac{1}{2}\left(-\frac{1}{u}\right) + C_2 = -\frac{1}{2(x^2-4x+6)} + C_2$$

$$I_3 = 3\int \frac{1}{\left[(x-2)^2 + \left(\sqrt{2}\right)^2\right]^2}\,dx = 3\int \frac{1}{[2(\tan^2\theta+1)]^2}\sqrt{2}\sec^2\theta\,d\theta \qquad \left[\begin{matrix} x - 2 = \sqrt{2}\tan\theta, \\ dx = \sqrt{2}\sec^2\theta\,d\theta \end{matrix}\right]$$
$$= \frac{3\sqrt{2}}{4}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \frac{3\sqrt{2}}{4}\int \cos^2\theta\,d\theta = \frac{3\sqrt{2}}{4}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$
$$= \frac{3\sqrt{2}}{8}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C_3 = \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3\sqrt{2}}{8}\left(\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta\right) + C_3$$
$$= \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3\sqrt{2}}{8}\cdot\frac{x-2}{\sqrt{x^2-4x+6}}\cdot\frac{\sqrt{2}}{\sqrt{x^2-4x+6}} + C_3$$
$$= \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3(x-2)}{4(x^2-4x+6)} + C_3$$



So $I = I_1 + I_2 + I_3 \qquad [C = C_1 + C_2 + C_3]$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{-1}{2(x^2-4x+6)} + \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3(x-2)}{4(x^2-4x+6)} + C$$
$$= \left(\frac{4\sqrt{2}}{8} + \frac{3\sqrt{2}}{8}\right)\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3(x-2)-2}{4(x^2-4x+6)} + C = \frac{7\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3x-8}{4(x^2-4x+6)} + C$$

38. $\dfrac{x^3+2x^2+3x-2}{(x^2+2x+2)^2} = \dfrac{Ax+B}{x^2+2x+2} + \dfrac{Cx+D}{(x^2+2x+2)^2} \quad \Rightarrow$

$x^3+2x^2+3x-2 = (Ax+B)(x^2+2x+2)+Cx+D \quad \Rightarrow$

$x^3+2x^2+3x-2 = Ax^3+(2A+B)x^2+(2A+2B+C)x+2B+D.$

So $A=1, 2A+B=2 \quad \Rightarrow \quad B=0, 2A+2B+C=3 \quad \Rightarrow \quad C=1$, and $2B+D=-2 \quad \Rightarrow \quad D=-2$. Thus,

$$I = \int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx = \int \left(\frac{x}{x^2+2x+2} + \frac{x-2}{(x^2+2x+2)^2}\right) dx$$

$$= \int \frac{x+1}{x^2+2x+2}\,dx + \int \frac{-1}{x^2+2x+2}\,dx + \int \frac{x+1}{(x^2+2x+2)^2}\,dx + \int \frac{-3}{(x^2+2x+2)^2}\,dx$$

$$= I_1+I_2+I_3+I_4.$$

$$I_1 = \int \frac{x+1}{x^2+2x+2}\,dx = \int \frac{1}{u}\left(\frac{1}{2}\,du\right) \quad \begin{bmatrix} u = x^2+2x+2, \\ du = 2(x+1)\,dx \end{bmatrix} \quad = \frac{1}{2}\ln\left|x^2+2x+2\right| + C_1$$

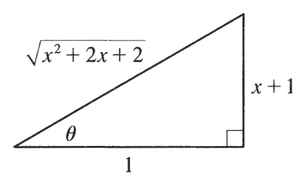
$$I_2 = -\int \frac{1}{(x+1)^2+1}\,dx = -\frac{1}{1}\tan^{-1}\left(\frac{x+1}{1}\right) + C_2 = -\tan^{-1}(x+1)+C_2$$

$$I_3 = \int \frac{x+1}{(x^2+2x+2)^2}\,dx = \int \frac{1}{u^2}\left(\frac{1}{2}\,du\right) = -\frac{1}{2u}+C_3 = -\frac{1}{2(x^2+2x+2)}+C_3$$

$$I_4 = -3\int \frac{1}{[(x+1)^2+1]^2}\,dx = -3\int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\,d\theta \quad \begin{bmatrix} x+1 = 1\tan\theta, \\ dx = \sec^2\theta\,d\theta \end{bmatrix}$$

$$= -3\int \frac{1}{\sec^2\theta}\,d\theta = -3\int \cos^2\theta\,d\theta = -\frac{3}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= -\tfrac{3}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C_4 = -\tfrac{3}{2}\theta - \tfrac{3}{2}\left(\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta\right) + C_4$$



$$= -\frac{3}{2}\tan^{-1}\left(\frac{x+1}{1}\right) - \frac{3}{2}\cdot\frac{x+1}{\sqrt{x^2+2x+2}}\cdot\frac{1}{\sqrt{x^2+2x+2}} + C_4$$

$$= -\frac{3}{2}\tan^{-1}(x+1) - \frac{3(x+1)}{2(x^2+2x+2)} + C_4$$

So $\quad I = I_1+I_2+I_3+I_4 \qquad [C = C_1+C_2+C_3+C_4]$

$$= \frac{1}{2}\ln(x^2+2x+2) - \tan^{-1}(x+1) - \frac{1}{2(x^2+2x+2)} - \frac{3}{2}\tan^{-1}(x+1) - \frac{3(x+1)}{2(x^2+2x+2)} + C$$

$$= \frac{1}{2}\ln(x^2+2x+2) - \frac{5}{2}\tan^{-1}(x+1) - \frac{3x+4}{2(x^2+2x+2)} + C$$

39. Let $u = \sqrt{x+1}$. Then $x = u^2-1$, $dx = 2u\,du \quad \Rightarrow$

$$\int \frac{dx}{x\sqrt{x+1}} = \int \frac{2u\,du}{(u^2-1)\,u} = 2\int \frac{du}{u^2-1} = \ln\left|\frac{u-1}{u+1}\right| + C = \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C.$$

40. Let $u = \sqrt{x+3}$, so $u^2 = x + 3$ and $2u\,du = dx$. Then

$$\int \frac{dx}{2\sqrt{x+3}+x} = \int \frac{2u\,du}{2u+(u^2-3)} = \int \frac{2u}{u^2+2u-3}\,du = \int \frac{2u}{(u+3)(u-1)}\,du. \text{ Now}$$

$$\frac{2u}{(u+3)(u-1)} = \frac{A}{u+3} + \frac{B}{u-1} \quad \Rightarrow \quad 2u = A(u-1) + B(u+3). \text{ Setting } u = 1 \text{ gives } 2 = 4B, \text{ so } B = \tfrac{1}{2}.$$

Setting $u = -3$ gives $-6 = -4A$, so $A = \frac{3}{2}$. Thus,

$$\int \frac{2u}{(u+3)(u-1)}\,du = \int \left(\frac{\frac{3}{2}}{u+3} + \frac{\frac{1}{2}}{u-1}\,du \right)$$

$$= \tfrac{3}{2} \ln|u+3| + \tfrac{1}{2} \ln|u-1| + C = \tfrac{3}{2} \ln\left(\sqrt{x+3}+3\right) + \tfrac{1}{2} \ln\left|\sqrt{x+3}-1\right| + C$$

41. Let $u = \sqrt{x}$, so $u^2 = x$ and $dx = 2u\,du$. Thus,

$$\int_9^{16} \frac{\sqrt{x}}{x-4}\,dx = \int_3^4 \frac{u}{u^2-4} 2u\,du = 2\int_3^4 \frac{u^2}{u^2-4}\,du = 2\int_3^4 \left(1 + \frac{4}{u^2-4}\right) du \qquad \text{[by long division]}$$

$$= 2 + 8\int_3^4 \frac{du}{(u+2)(u-2)} \quad (\star)$$

Multiply $\dfrac{1}{(u+2)(u-2)} = \dfrac{A}{u+2} + \dfrac{B}{u-2}$ by $(u+2)(u-2)$ to get $1 = A(u-2) + B(u+2)$. Equating coefficients we get $A + B = 0$ and $-2A + 2B = 1$. Solving gives us $B = \frac{1}{4}$ and $A = -\frac{1}{4}$, so $\dfrac{1}{(u+2)(u-2)} = \dfrac{-1/4}{u+2} + \dfrac{1/4}{u-2}$ and $(\star)$ is

$$2 + 8\int_3^4 \left(\frac{-1/4}{u+2} + \frac{1/4}{u-2}\right) du = 2 + 8\left[-\tfrac{1}{4}\ln|u+2| + \tfrac{1}{4}\ln|u-2|\right]_3^4 = 2 + \left[2\ln|u-2| - 2\ln|u+2|\right]_3^4$$

$$= 2 + 2\left[\ln\left|\frac{u-2}{u+2}\right|\right]_3^4 = 2 + 2\left(\ln\tfrac{2}{6} - \ln\tfrac{1}{5}\right) = 2 + 2\ln\tfrac{2/6}{1/5}$$

$$= 2 + 2\ln\tfrac{5}{3} \;\text{ or }\; 2 + \ln\left(\tfrac{5}{3}\right)^2 = 2 + \ln\tfrac{25}{9}$$

42. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2\,du \quad \Rightarrow$

$$\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx = \int_0^1 \frac{3u^2\,du}{1+u} = \int_0^1 \left(3u - 3 + \frac{3}{1+u}\right) du = \left[\tfrac{3}{2}u^2 - 3u + 3\ln(1+u)\right]_0^1 = 3\left(\ln 2 - \tfrac{1}{2}\right).$$

43. Let $u = \sqrt[3]{x^2+1}$. Then $x^2 = u^3 - 1$, $2x\,dx = 3u^2\,du \quad \Rightarrow$

$$\int \frac{x^3\,dx}{\sqrt[3]{x^2+1}} = \int \frac{(u^3-1)\frac{3}{2}u^2\,du}{u} = \frac{3}{2}\int (u^4 - u)\,du$$

$$= \tfrac{3}{10}u^5 - \tfrac{3}{4}u^2 + C = \tfrac{3}{10}(x^2+1)^{5/3} - \tfrac{3}{4}(x^2+1)^{2/3} + C$$

44. Let $u = \sqrt{x}$. Then $x = u^2$, $dx = 2u\,du \quad \Rightarrow$

$$\int_{1/3}^3 \frac{\sqrt{x}}{x^2+x}\,dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u \cdot 2u\,du}{u^4+u^2} = 2\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{u^2+1} = 2\left[\tan^{-1} u\right]_{1/\sqrt{3}}^{\sqrt{3}} = 2\left(\tfrac{\pi}{3} - \tfrac{\pi}{6}\right) = \tfrac{\pi}{3}.$$

45. If we were to substitute $u = \sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution $u = \sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u = \sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3.)

Let $u = \sqrt[6]{x}$. Then $x = u^6$, so $dx = 6u^5\,du$ and $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$. Thus,

$$\begin{aligned}\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} &= \int \frac{6u^5\,du}{u^3 - u^2} = 6\int \frac{u^5}{u^2(u-1)}\,du = 6\int \frac{u^3}{u-1}\,du \\ &= 6\int \left(u^2 + u + 1 + \frac{1}{u-1}\right) du \qquad \text{[by long division]} \\ &= 6\left(\tfrac{1}{3}u^3 + \tfrac{1}{2}u^2 + u + \ln|u-1|\right) + C = 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln\left|\sqrt[6]{x} - 1\right| + C\end{aligned}$$

46. Let $u = \sqrt{1+\sqrt{x}}$, so that $u^2 = 1 + \sqrt{x}$, $x = (u^2-1)^2$, and $dx = 2(u^2-1)\cdot 2u\,du = 4u(u^2-1)\,du$. Then

$\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx = \int \frac{u}{(u^2-1)^2}\cdot 4u(u^2-1)\,du = \int \frac{4u^2}{u^2-1}\,du = \int\left(4 + \frac{4}{u^2-1}\right) du$. Now

$\displaystyle\frac{4}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 4 = A(u-1) + B(u+1)$. Setting $u = 1$ gives $4 = 2B$, so $B = 2$. Setting $u = -1$ gives $4 = -2A$, so $A = -2$. Thus,

$$\begin{aligned}\int\left(4 + \frac{4}{u^2-1}\right) du &= \int\left(4 - \frac{2}{u+1} + \frac{2}{u-1}\right) du = 4u - 2\ln|u+1| + 2\ln|u-1| + C \\ &= 4\sqrt{1+\sqrt{x}} - 2\ln\left(\sqrt{1+\sqrt{x}} + 1\right) + 2\ln\left(\sqrt{1+\sqrt{x}} - 1\right) + C\end{aligned}$$

47. Let $u = e^x$. Then $x = \ln u$, $dx = \dfrac{du}{u} \Rightarrow$

$$\begin{aligned}\int \frac{e^{2x}\,dx}{e^{2x} + 3e^x + 2} &= \int \frac{u^2\,(du/u)}{u^2 + 3u + 2} = \int \frac{u\,du}{(u+1)(u+2)} = \int\left[\frac{-1}{u+1} + \frac{2}{u+2}\right] du \\ &= 2\ln|u+2| - \ln|u+1| + C = \ln\frac{(e^x+2)^2}{e^x+1} + C\end{aligned}$$

48. Let $u = \sin x$. Then $du = \cos x\,dx \Rightarrow$

$$\int \frac{\cos x\,dx}{\sin^2 x + \sin x} = \int \frac{du}{u^2 + u} = \int \frac{du}{u(u+1)} = \int\left[\frac{1}{u} - \frac{1}{u+1}\right] du = \ln\left|\frac{u}{u+1}\right| + C = \ln\left|\frac{\sin x}{1 + \sin x}\right| + C.$$

49. Let $u = \tan t$, so that $du = \sec^2 t\,dt$. Then $\displaystyle\int \frac{\sec^2 t}{\tan^2 t + 3\tan t + 2}\,dt = \int \frac{1}{u^2 + 3u + 2}\,du = \int \frac{1}{(u+1)(u+2)}\,du.$

Now $\displaystyle\frac{1}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2} \Rightarrow 1 = A(u+2) + B(u+1).$

Setting $u = -2$ gives $1 = -B$, so $B = -1$. Setting $u = -1$ gives $1 = A$.

Thus, $\displaystyle\int \frac{1}{(u+1)(u+2)}\,du = \int\left(\frac{1}{u+1} - \frac{1}{u+2}\right) du = \ln|u+1| - \ln|u+2| + C = \ln|\tan t + 1| - \ln|\tan t + 2| + C.$

50. Let $u = e^x$, so that $du = e^x\,dx$. Then $\displaystyle\int \frac{e^x}{(e^x-2)(e^{2x}+1)}\,dx = \int \frac{1}{(u-2)(u^2+1)}\,du$. Now

$\displaystyle\frac{1}{(u-2)(u^2+1)} = \frac{A}{u-2} + \frac{Bu+C}{u^2+1} \quad\Rightarrow\quad 1 = A(u^2+1) + (Bu+C)(u-2)$. Setting $u = 2$ gives $1 = 5A$, so $A = \frac{1}{5}$.

Setting $u = 0$ gives $1 = \frac{1}{5} - 2C$, so $C = -\frac{2}{5}$. Comparing coefficients of u^2 gives $0 = \frac{1}{5} + B$, so $B = -\frac{1}{5}$. Thus,

$$\begin{aligned}\int \frac{1}{(u-2)(u^2+1)}\,du &= \int \left(\frac{\frac{1}{5}}{u-2} + \frac{-\frac{1}{5}u - \frac{2}{5}}{u^2+1}\right) du = \frac{1}{5}\int \frac{1}{u-2}\,du - \frac{1}{5}\int \frac{u}{u^2+1}\,du - \frac{2}{5}\int \frac{1}{u^2+1}\,du \\ &= \tfrac{1}{5}\ln|u-2| - \tfrac{1}{5}\cdot\tfrac{1}{2}\ln|u^2+1| - \tfrac{2}{5}\tan^{-1}u + C \\ &= \tfrac{1}{5}\ln|e^x-2| - \tfrac{1}{10}\ln(e^{2x}+1) - \tfrac{2}{5}\tan^{-1}e^x + C\end{aligned}$$

51. Let $u = \ln(x^2 - x + 2)$, $dv = dx$. Then $du = \dfrac{2x-1}{x^2-x+2}\,dx$, $v = x$, and (by integration by parts)

$$\begin{aligned}\int \ln(x^2-x+2)\,dx &= x\ln(x^2-x+2) - \int \frac{2x^2-x}{x^2-x+2}\,dx = x\ln(x^2-x+2) - \int \left(2 + \frac{x-4}{x^2-x+2}\right) dx \\ &= x\ln(x^2-x+2) - 2x - \int \frac{\frac{1}{2}(2x-1)}{x^2-x+2}\,dx + \frac{7}{2}\int \frac{dx}{(x-\frac{1}{2})^2 + \frac{7}{4}} \\ &= x\ln(x^2-x+2) - 2x - \frac{1}{2}\ln(x^2-x+2) + \frac{7}{2}\int \frac{\frac{\sqrt{7}}{2}\,du}{\frac{7}{4}(u^2+1)} \qquad \left[\begin{array}{c}\text{where } x - \frac{1}{2} = \frac{\sqrt{7}}{2}u, \\ dx = \frac{\sqrt{7}}{2}\,du, \\ (x-\frac{1}{2})^2 + \frac{7}{4} = \frac{7}{4}(u^2+1)\end{array}\right] \\ &= (x - \tfrac{1}{2})\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}u + C \\ &= (x - \tfrac{1}{2})\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}\frac{2x-1}{\sqrt{7}} + C\end{aligned}$$

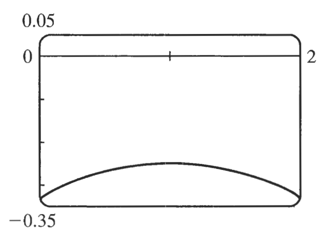
52. Let $u = \tan^{-1}x$, $dv = x\,dx \quad\Rightarrow\quad du = dx/(1+x^2)$, $v = \frac{1}{2}x^2$.

Then $\displaystyle\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$. To evaluate the last integral, use long division or observe that

$\displaystyle\int \frac{x^2}{1+x^2}\,dx = \int \frac{(1+x^2)-1}{1+x^2}\,dx = \int 1\,dx - \int \frac{1}{1+x^2}\,dx = x - \tan^{-1}x + C_1$. So

$\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x + C_1) = \frac{1}{2}(x^2\tan^{-1}x + \tan^{-1}x - x) + C$.

53.



From the graph, we see that the integral will be negative, and we guess that the area is about the same as that of a rectangle with width 2 and height 0.3, so we estimate the integral to be $-(2\cdot 0.3) = -0.6$. Now

$\displaystyle\frac{1}{x^2-2x-3} = \frac{1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \quad\Leftrightarrow$

$1 = (A+B)x + A - 3B$, so $A = -B$ and $A - 3B = 1 \quad\Leftrightarrow\quad A = \frac{1}{4}$

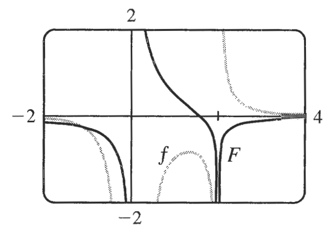
and $B = -\frac{1}{4}$, so the integral becomes

$$\begin{aligned}\int_0^2 \frac{dx}{x^2-2x-3} &= \frac{1}{4}\int_0^2 \frac{dx}{x-3} - \frac{1}{4}\int_0^2 \frac{dx}{x+1} = \frac{1}{4}\Big[\ln|x-3| - \ln|x+1|\Big]_0^2 = \frac{1}{4}\left[\ln\left|\frac{x-3}{x+1}\right|\right]_0^2 \\ &= \tfrac{1}{4}\left(\ln\tfrac{1}{3} - \ln 3\right) = -\tfrac{1}{2}\ln 3 \approx -0.55\end{aligned}$$

54. $\dfrac{1}{x^3-2x^2} = \dfrac{1}{x^2(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2} \quad \Rightarrow \quad 1 = (A+C)x^2 + (B-2A)x - 2B$, so $A + C = B - 2A = 0$ and

$-2B = 1 \quad \Rightarrow \quad B = -\frac{1}{2}$, $A = -\frac{1}{4}$, and $C = \frac{1}{4}$. So the general antiderivative of $\dfrac{1}{x^3-2x^2}$ is

$$\begin{aligned}\int \frac{dx}{x^3-2x^2} &= -\frac{1}{4}\int\frac{dx}{x} - \frac{1}{2}\int\frac{dx}{x^2} + \frac{1}{4}\int\frac{dx}{x-2}\\ &= -\tfrac{1}{4}\ln|x| - \tfrac{1}{2}(-1/x) + \tfrac{1}{4}\ln|x-2| + C\\ &= \frac{1}{4}\ln\left|\frac{x-2}{x}\right| + \frac{1}{2x} + C\end{aligned}$$



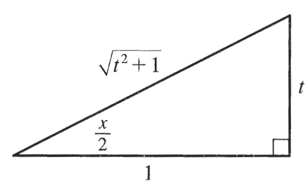
We plot this function with $C = 0$ on the same screen as $y = \dfrac{1}{x^3-2x^2}$.

55. $$\begin{aligned}\int \frac{dx}{x^2-2x} &= \int\frac{dx}{(x-1)^2-1} = \int\frac{du}{u^2-1} \quad [\text{put } u = x-1]\\ &= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C \quad [\text{by Equation 6}] \quad = \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C\end{aligned}$$

56. $$\begin{aligned}\int\frac{(2x+1)\,dx}{4x^2+12x-7} &= \frac{1}{4}\int\frac{(8x+12)\,dx}{4x^2+12x-7} - \int\frac{2\,dx}{(2x+3)^2-16}\\ &= \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \int\frac{du}{u^2-16} \quad [\text{put } u = 2x+3]\\ &= \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \tfrac{1}{8}\ln|(u-4)/(u+4)| + C \quad [\text{by Equation 6}]\\ &= \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \tfrac{1}{8}\ln|(2x-1)/(2x+7)| + C\end{aligned}$$

57. (a) If $t = \tan\left(\dfrac{x}{2}\right)$, then $\dfrac{x}{2} = \tan^{-1} t$. The figure gives

$\cos\left(\dfrac{x}{2}\right) = \dfrac{1}{\sqrt{1+t^2}}$ and $\sin\left(\dfrac{x}{2}\right) = \dfrac{t}{\sqrt{1+t^2}}$.



(b) $$\begin{aligned}\cos x &= \cos\left(2\cdot\frac{x}{2}\right) = 2\cos^2\left(\frac{x}{2}\right) - 1\\ &= 2\left(\frac{1}{\sqrt{1+t^2}}\right)^2 - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}\end{aligned}$$

(c) $\dfrac{x}{2} = \arctan t \quad \Rightarrow \quad x = 2\arctan t \quad \Rightarrow \quad dx = \dfrac{2}{1+t^2}\,dt$

58. Let $t = \tan(x/2)$. Then, using Exercise 57, $dx = \dfrac{2}{1+t^2}\,dt$, $\sin x = \dfrac{2t}{1+t^2} \quad \Rightarrow$

$$\begin{aligned}\int\frac{dx}{3-5\sin x} &= \int\frac{2\,dt/(1+t^2)}{3-10t/(1+t^2)} = \int\frac{2\,dt}{3(1+t^2)-10t} = 2\int\frac{dt}{3t^2-10t+3}\\ &= \frac{1}{4}\int\left[\frac{1}{t-3} - \frac{3}{3t-1}\right]dt = \tfrac{1}{4}(\ln|t-3| - \ln|3t-1|) + C = \frac{1}{4}\ln\left|\frac{\tan(x/2)-3}{3\tan(x/2)-1}\right| + C\end{aligned}$$

59. Let $t = \tan(x/2)$. Then, using the expressions in Exercise 57, we have

$$\int \frac{1}{3\sin x - 4\cos x}\,dx = \int \frac{1}{3\left(\dfrac{2t}{1+t^2}\right) - 4\left(\dfrac{1-t^2}{1+t^2}\right)}\,\frac{2\,dt}{1+t^2} = 2\int \frac{dt}{3(2t) - 4(1-t^2)} = \int \frac{dt}{2t^2+3t-2}$$

$$= \int \frac{dt}{(2t-1)(t+2)} = \int \left[\frac{2}{5}\frac{1}{2t-1} - \frac{1}{5}\frac{1}{t+2}\right] dt \qquad \text{[using partial fractions]}$$

$$= \tfrac{1}{5}\Big[\ln|2t-1| - \ln|t+2|\Big] + C = \frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right| + C = \frac{1}{5}\ln\left|\frac{2\tan(x/2)-1}{\tan(x/2)+2}\right| + C$$

60. Let $t = \tan(x/2)$. Then, by Exercise 57,

$$\int_{\pi/3}^{\pi/2} \frac{dx}{1+\sin x - \cos x} = \int_{1/\sqrt{3}}^{1} \frac{2\,dt/(1+t^2)}{1 + 2t/(1+t^2) - (1-t^2)/(1+t^2)} = \int_{1/\sqrt{3}}^{1} \frac{2\,dt}{1+t^2+2t-1+t^2}$$

$$= \int_{1/\sqrt{3}}^{1} \left[\frac{1}{t} - \frac{1}{t+1}\right] dt = \Big[\ln t - \ln(t+1)\Big]_{1/\sqrt{3}}^{1} = \ln\frac{1}{2} - \ln\frac{1}{\sqrt{3}+1} = \ln\frac{\sqrt{3}+1}{2}$$

61. Let $t = \tan(x/2)$. Then, by Exercise 57,

$$\int_0^{\pi/2} \frac{\sin 2x}{2+\cos x}\,dx = \int_0^{\pi/2} \frac{2\sin x\,\cos x}{2+\cos x}\,dx = \int_0^1 \frac{2\cdot\dfrac{2t}{1+t^2}\cdot\dfrac{1-t^2}{1+t^2}}{2+\dfrac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,dt = \int_0^1 \frac{\dfrac{8t(1-t^2)}{(1+t^2)^2}}{2(1+t^2)+(1-t^2)}\,dt$$

$$= \int_0^1 8t\cdot\frac{1-t^2}{(t^2+3)(t^2+1)^2}\,dt = I$$

If we now let $u = t^2$, then $\dfrac{1-t^2}{(t^2+3)(t^2+1)^2} = \dfrac{1-u}{(u+3)(u+1)^2} = \dfrac{A}{u+3} + \dfrac{B}{u+1} + \dfrac{C}{(u+1)^2} \Rightarrow$

$1 - u = A(u+1)^2 + B(u+3)(u+1) + C(u+3)$. Set $u = -1$ to get $2 = 2C$, so $C = 1$. Set $u = -3$ to get $4 = 4A$, so $A = 1$. Set $u = 0$ to get $1 = 1 + 3B + 3$, so $B = -1$. So

$$I = \int_0^1 \left[\frac{8t}{t^2+3} - \frac{8t}{t^2+1} + \frac{8t}{(t^2+1)^2}\right] dt = \left[4\ln(t^2+3) - 4\ln(t^2+1) - \frac{4}{t^2+1}\right]_0^1$$

$$= (4\ln 4 - 4\ln 2 - 2) - (4\ln 3 - 0 - 4) = 8\ln 2 - 4\ln 2 - 4\ln 3 + 2 = 4\ln\tfrac{2}{3} + 2$$

62. $\dfrac{1}{x^3+x} = \dfrac{1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)x$. Set $x = 0$ to get $1 = A$. So

$1 = (1+B)x^2 + Cx + 1 \Rightarrow B + 1 = 0 \quad [B = -1]$ and $C = 0$. Thus, the area is

$$\int_1^2 \frac{1}{x^3+x}\,dx = \int_1^2 \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \Big[\ln|x| - \tfrac{1}{2}\ln|x^2+1|\Big]_1^2 = \left(\ln 2 - \tfrac{1}{2}\ln 5\right) - \left(0 - \tfrac{1}{2}\ln 2\right)$$

$$= \tfrac{3}{2}\ln 2 - \tfrac{1}{2}\ln 5 \quad \left[\text{or } \tfrac{1}{2}\ln\tfrac{8}{5}\right]$$

63. By long division, $\dfrac{x^2+1}{3x-x^2} = -1 + \dfrac{3x+1}{3x-x^2}$. Now

$\dfrac{3x+1}{3x-x^2} = \dfrac{3x+1}{x(3-x)} = \dfrac{A}{x} + \dfrac{B}{3-x} \quad\Rightarrow\quad 3x+1 = A(3-x)+Bx$. Set $x=3$ to get $10=3B$, so $B=\frac{10}{3}$. Set $x=0$ to get $1=3A$, so $A=\frac{1}{3}$. Thus, the area is

$$\int_1^2 \frac{x^2+1}{3x-x^2}\,dx = \int_1^2 \left(-1+\frac{\frac{1}{3}}{x}+\frac{\frac{10}{3}}{3-x}\right)dx = \left[-x+\tfrac{1}{3}\ln|x|-\tfrac{10}{3}\ln|3-x|\right]_1^2$$
$$= \left(-2+\tfrac{1}{3}\ln 2-0\right)-\left(-1+0-\tfrac{10}{3}\ln 2\right) = -1+\tfrac{11}{3}\ln 2$$

64. (a) We use disks, so the volume is $V = \pi\displaystyle\int_0^1 \left[\frac{1}{x^2+3x+2}\right]^2 dx = \pi\int_0^1 \frac{dx}{(x+1)^2(x+2)^2}$. To evaluate the integral, we use partial fractions: $\dfrac{1}{(x+1)^2(x+2)^2} = \dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{C}{x+2}+\dfrac{D}{(x+2)^2} \quad\Rightarrow$

$1 = A(x+1)(x+2)^2+B(x+2)^2+C(x+1)^2(x+2)+D(x+1)^2$. We set $x=-1$, giving $B=1$, then set $x=-2$, giving $D=1$. Now equating coefficients of x^3 gives $A=-C$, and then equating constants gives $1 = 4A+4+2(-A)+1 \quad\Rightarrow\quad A=-2 \quad\Rightarrow\quad C=2$. So the expression becomes

$$V = \pi\int_0^1 \left[\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{2}{(x+2)}+\frac{1}{(x+2)^2}\right]dx = \pi\left[2\ln\left|\frac{x+2}{x+1}\right|-\frac{1}{x+1}-\frac{1}{x+2}\right]_0^1$$
$$= \pi\left[\left(2\ln\tfrac{3}{2}-\tfrac{1}{2}-\tfrac{1}{3}\right)-\left(2\ln 2-1-\tfrac{1}{2}\right)\right] = \pi\left(2\ln\frac{3/2}{2}+\frac{2}{3}\right) = \pi\left(\tfrac{2}{3}+\ln\tfrac{9}{16}\right)$$

(b) In this case, we use cylindrical shells, so the volume is $V = 2\pi\displaystyle\int_0^1 \frac{x\,dx}{x^2+3x+2} = 2\pi\int_0^1 \frac{x\,dx}{(x+1)(x+2)}$. We use partial fractions to simplify the integrand: $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1}+\dfrac{B}{x+2} \quad\Rightarrow\quad x = (A+B)x+2A+B$. So $A+B=1$ and $2A+B=0 \quad\Rightarrow\quad A=-1$ and $B=2$. So the volume is

$$2\pi\int_0^1\left[\frac{-1}{x+1}+\frac{2}{x+2}\right]dx = 2\pi\Big[-\ln|x+1|+2\ln|x+2|\Big]_0^1$$
$$= 2\pi(-\ln 2+2\ln 3+\ln 1-2\ln 2) = 2\pi(2\ln 3-3\ln 2) = 2\pi\ln\tfrac{9}{8}$$

65. $\dfrac{P+S}{P\,[(r-1)P-S]} = \dfrac{A}{P}+\dfrac{B}{(r-1)P-S} \quad\Rightarrow\quad P+S = A\,[(r-1)P-S]+BP = [(r-1)A+B]\,P-AS \quad\Rightarrow$

$(r-1)A+B=1,\ -A=1 \quad\Rightarrow\quad A=-1,\ B=r$. Now

$$t = \int\frac{P+S}{P\,[(r-1)P-S]}\,dP = \int\left[\frac{-1}{P}+\frac{r}{(r-1)P-S}\right]dP = -\int\frac{dP}{P}+\frac{r}{r-1}\int\frac{r-1}{(r-1)P-S}\,dP$$

so $t = -\ln P+\dfrac{r}{r-1}\ln|(r-1)P-S|+C$. Here $r=0.10$ and $S=900$, so

$t = -\ln P+\frac{0.1}{-0.9}\ln|-0.9P-900|+C = -\ln P-\frac{1}{9}\ln(|-1|\,|0.9P+900|) = -\ln P-\frac{1}{9}\ln(0.9P+900)+C$.

When $t=0$, $P=10{,}000$, so $0 = -\ln 10{,}000-\frac{1}{9}\ln(9900)+C$. Thus, $C = \ln 10{,}000+\frac{1}{9}\ln 9900$ [≈ 10.2326], so our equation becomes

$$t = \ln 10{,}000 - \ln P + \tfrac{1}{9}\ln 9900 - \tfrac{1}{9}\ln(0.9P + 900) = \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{9900}{0.9P + 900}$$
$$= \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{1100}{0.1P + 100} = \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{11{,}000}{P + 1000}$$

66. If we subtract and add $2x^2$, we get

$$x^4 + 1 = x^4 + 2x^2 + 1 - 2x^2 = (x^2+1)^2 - 2x^2 = (x^2+1)^2 - (\sqrt{2}\,x)^2$$
$$= \left[(x^2+1) - \sqrt{2}\,x\right]\left[(x^2+1) + \sqrt{2}\,x\right] = \left(x^2 - \sqrt{2}\,x + 1\right)\left(x^2 + \sqrt{2}\,x + 1\right)$$

So we can decompose $\dfrac{1}{x^4+1} = \dfrac{Ax+B}{x^2+\sqrt{2}\,x+1} + \dfrac{Cx+D}{x^2-\sqrt{2}\,x+1} \quad\Rightarrow$

$1 = (Ax+B)\left(x^2 - \sqrt{2}\,x + 1\right) + (Cx+D)\left(x^2 + \sqrt{2}\,x + 1\right)$. Setting the constant terms equal gives $B + D = 1$, then from the coefficients of x^3 we get $A + C = 0$. Now from the coefficients of x we get $A + C + (B - D)\sqrt{2} = 0 \quad\Leftrightarrow$ $[(1-D) - D]\sqrt{2} = 0 \;\Rightarrow\; D = \frac{1}{2} \;\Rightarrow\; B = \frac{1}{2}$, and finally, from the coefficients of x^2 we get $\sqrt{2}\,(C - A) + B + D = 0 \;\Rightarrow\; C - A = -\frac{1}{\sqrt{2}} \;\Rightarrow\; C = -\frac{\sqrt{2}}{4}$ and $A = \frac{\sqrt{2}}{4}$. So we rewrite the integrand, splitting the terms into forms which we know how to integrate:

$$\frac{1}{x^4+1} = \frac{\frac{\sqrt{2}}{4}x + \frac{1}{2}}{x^2+\sqrt{2}\,x+1} + \frac{-\frac{\sqrt{2}}{4}x + \frac{1}{2}}{x^2-\sqrt{2}\,x+1} = \frac{1}{4\sqrt{2}}\left[\frac{2x + 2\sqrt{2}}{x^2+\sqrt{2}\,x+1} - \frac{2x - 2\sqrt{2}}{x^2-\sqrt{2}\,x+1}\right]$$
$$= \frac{\sqrt{2}}{8}\left[\frac{2x+\sqrt{2}}{x^2+\sqrt{2}\,x+1} - \frac{2x-\sqrt{2}}{x^2-\sqrt{2}\,x+1}\right] + \frac{1}{4}\left[\frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} + \frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}\right]$$

Now we integrate: $\displaystyle\int \frac{dx}{x^4+1} = \frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}\,x+1}{x^2-\sqrt{2}\,x+1}\right) + \frac{\sqrt{2}}{4}\left[\tan^{-1}\left(\sqrt{2}\,x+1\right) + \tan^{-1}\left(\sqrt{2}\,x-1\right)\right] + C.$

67. (a) In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to obtain

$$f(x) = \frac{24{,}110/4879}{5x+2} - \frac{668/323}{2x+1} - \frac{9438/80{,}155}{3x-7} + \frac{(22{,}098x + 48{,}935)/260{,}015}{x^2+x+5}$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

(b) $\displaystyle\int f(x)\,dx = \frac{24{,}110}{4879}\cdot\frac{1}{5}\ln|5x+2| - \frac{668}{323}\cdot\frac{1}{2}\ln|2x+1| - \frac{9438}{80{,}155}\cdot\frac{1}{3}\ln|3x-7|$

$$+\frac{1}{260{,}015}\int\frac{22{,}098\left(x+\frac{1}{2}\right) + 37{,}886}{\left(x+\frac{1}{2}\right)^2 + \frac{19}{4}}\,dx + C$$
$$= \tfrac{24{,}110}{4879}\cdot\tfrac{1}{5}\ln|5x+2| - \tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1| - \tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7|$$
$$+\tfrac{1}{260{,}015}\left[22{,}098\cdot\tfrac{1}{2}\ln\left(x^2+x+5\right) + 37{,}886\cdot\sqrt{\tfrac{4}{19}}\tan^{-1}\left(\tfrac{1}{\sqrt{19/4}}\left(x+\tfrac{1}{2}\right)\right)\right] + C$$
$$= \tfrac{4822}{4879}\ln|5x+2| - \tfrac{334}{323}\ln|2x+1| - \tfrac{3146}{80{,}155}\ln|3x-7| + \tfrac{11{,}049}{260{,}015}\ln\left(x^2+x+5\right)$$
$$+\tfrac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\left[\tfrac{1}{\sqrt{19}}(2x+1)\right] + C$$

[continued]

Using a CAS, we get

$$\frac{4822\ln(5x+2)}{4879} - \frac{334\ln(2x+1)}{323} - \frac{3146\ln(3x-7)}{80{,}155} + \frac{11{,}049\ln(x^2+x+5)}{260{,}015} + \frac{3988\sqrt{19}}{260{,}015}\tan^{-1}\left[\frac{\sqrt{19}}{19}(2x+1)\right]$$

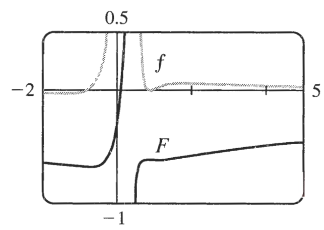
The main difference in this answer is that the absolute value signs and the constant of integration have been omitted. Also, the fractions have been reduced and the denominators rationalized.

68. (a) In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to get

$$f(x) = \frac{5828/1815}{(5x-2)^2} - \frac{59{,}096/19{,}965}{5x-2} + \frac{2(2843x+816)/3993}{2x^2+1} + \frac{(313x-251)/363}{(2x^2+1)^2}.$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

(b) As we saw in Exercise 67, computer algebra systems omit the absolute value signs in $\int (1/y)\,dy = \ln|y|$. So we use the CAS to integrate the expression in part (a) and add the necessary absolute value signs and constant of integration to get



$$\int f(x)\,dx = -\frac{5828}{9075(5x-2)} - \frac{59{,}096\ln|5x-2|}{99{,}825} + \frac{2843\ln(2x^2+1)}{7986} + \frac{503}{15{,}972}\sqrt{2}\tan^{-1}(\sqrt{2}\,x) - \frac{1}{2904}\frac{1004x+626}{2x^2+1} + C$$

(c) From the graph, we see that f goes from negative to positive at $x \approx -0.78$, then back to negative at $x \approx 0.8$, and finally back to positive at $x = 1$. Also, $\lim_{x\to 0.4} f(x) = \infty$. So we see (by the First Derivative Test) that $\int f(x)\,dx$ has minima at $x \approx -0.78$ and $x = 1$, and a maximum at $x \approx 0.80$, and that $\int f(x)\,dx$ is unbounded as $x \to 0.4$. Note also that just to the right of $x = 0.4$, f has large values, so $\int f(x)\,dx$ increases rapidly, but slows down as f drops toward 0. $\int f(x)\,dx$ decreases from about 0.8 to 1, then increases slowly since f stays small and positive.

69. There are only finitely many values of x where $Q(x) = 0$ (assuming that Q is not the zero polynomial). At all other values of x, $F(x)/Q(x) = G(x)/Q(x)$, so $F(x) = G(x)$. In other words, the values of F and G agree at all except perhaps finitely many values of x. By continuity of F and G, the polynomials F and G must agree at those values of x too.

More explicitly: if a is a value of x such that $Q(a) = 0$, then $Q(x) \neq 0$ for all x sufficiently close to a. Thus,

$$\begin{aligned} F(a) &= \lim_{x\to a} F(x) && \text{[by continuity of } F\text{]} \\ &= \lim_{x\to a} G(x) && \text{[whenever } Q(x) \neq 0\text{]} \\ &= G(a) && \text{[by continuity of } G\text{]} \end{aligned}$$

70. Let $f(x) = ax^2 + bx + c$. We calculate the partial fraction decomposition of $\dfrac{f(x)}{x^2(x+1)^3}$. Since $f(0) = 1$, we must have $c = 1$, so $\dfrac{f(x)}{x^2(x+1)^3} = \dfrac{ax^2 + bx + 1}{x^2(x+1)^3} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2} + \dfrac{E}{(x+1)^3}$. Now in order for the integral not to contain any logarithms (that is, in order for it to be a rational function), we must have $A = C = 0$, so $ax^2 + bx + 1 = B(x+1)^3 + Dx^2(x+1) + Ex^2$. Equating constant terms gives $B = 1$, then equating coefficients of x gives $3B = b \quad \Rightarrow \quad b = 3$. This is the quantity we are looking for, since $f'(0) = b$.