1. Let $u = \sin x$, so that $du = \cos x\,dx$. Then $\int \cos x(1+\sin^2 x)\,dx = \int (1+u^2)\,du = u + \frac{1}{3}u^3 + C = \sin x + \frac{1}{3}\sin^3 x + C$.

2. $$\int \frac{\sin^3 x}{\cos x}\,dx = \int \frac{\sin^2 x\,\sin x}{\cos x}\,dx = \int \frac{(1-\cos^2 x)\sin x}{\cos x}\,dx = \int \frac{1-u^2}{u}(-du) \quad \left[\begin{array}{l} u=\cos x \\ du = -\sin x\,dx \end{array}\right]$$
$$= \int \left(u - \tfrac{1}{u}\right)du = \tfrac{1}{2}u^2 - \ln|u| + C = \tfrac{1}{2}\cos^2 x - \ln|\cos x| + C$$

3. $$\int \frac{\sin x + \sec x}{\tan x}\,dx = \int \left(\frac{\sin x}{\tan x} + \frac{\sec x}{\tan x}\right)dx = \int (\cos x + \csc x)\,dx = \sin x + \ln|\csc x - \cot x| + C$$

4. $$\int \tan^3\theta\,d\theta = \int (\sec^2\theta - 1)\tan\theta\,d\theta = \int \tan\theta\,\sec^2\theta\,d\theta - \int \frac{\sin\theta}{\cos\theta}\,d\theta$$
$$= \int u\,du + \int \frac{dv}{v} \quad \left[\begin{array}{ll} u = \tan\theta, & v = \cos\theta, \\ du = \sec^2\theta\,d\theta & dv = -\sin\theta\,d\theta \end{array}\right]$$
$$= \tfrac{1}{2}u^2 + \ln|v| + C = \tfrac{1}{2}\tan^2\theta + \ln|\cos\theta| + C$$

5. $$\int_0^2 \frac{2t}{(t-3)^2}\,dt = \int_{-3}^{-1} \frac{2(u+3)}{u^2}\,du \quad \left[\begin{array}{l} u = t-3, \\ du = dt \end{array}\right] \quad = \int_{-3}^{-1}\left(\frac{2}{u} + \frac{6}{u^2}\right)du = \left[2\ln|u| - \frac{6}{u}\right]_{-3}^{-1}$$
$$= (2\ln 1 + 6) - (2\ln 3 + 2) = 4 - 2\ln 3 \text{ or } 4 - \ln 9$$

6. Let $u = x^2$. Then $du = 2x\,dx \;\Rightarrow\; \displaystyle\int \frac{x\,dx}{\sqrt{3-x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{3-u^2}} = \frac{1}{2}\sin^{-1}\frac{u}{\sqrt{3}} + C = \frac{1}{2}\sin^{-1}\frac{x^2}{\sqrt{3}} + C.$

7. Let $u = \arctan y$. Then $du = \dfrac{dy}{1+y^2} \;\Rightarrow\; \displaystyle\int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = \left[e^u\right]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}.$

8. $$\int x\csc x\cot x\,dx \quad \left[\begin{array}{ll} u = x, & dv = \csc x\cot x\,dx, \\ du = dx & v = -\csc x \end{array}\right] \quad = -x\csc x - \int(-\csc x)\,dx = -x\csc x + \ln|\csc x - \cot x| + C$$

9. $$\int_1^3 r^4\ln r\,dr \quad \left[\begin{array}{ll} u = \ln r, & dv = r^4\,dr, \\ du = \frac{dr}{r} & v = \frac{1}{5}r^5 \end{array}\right] \quad = \left[\tfrac{1}{5}r^5\ln r\right]_1^3 - \int_1^3 \tfrac{1}{5}r^4\,dr = \tfrac{243}{5}\ln 3 - 0 - \left[\tfrac{1}{25}r^5\right]_1^3$$
$$= \tfrac{243}{5}\ln 3 - \left(\tfrac{243}{25} - \tfrac{1}{25}\right) = \tfrac{243}{5}\ln 3 - \tfrac{242}{25}$$

10. $\dfrac{x-1}{x^2-4x-5} = \dfrac{x-1}{(x-5)(x+1)} = \dfrac{A}{x-5} + \dfrac{B}{x+1} \Rightarrow x-1 = A(x+1)+B(x-5)$. Setting $x=-1$ gives $-2=-6B$, so $B=\frac{1}{3}$. Setting $x=5$ gives $4=6A$, so $A=\frac{2}{3}$. Now

$$\int_0^4 \frac{x-1}{x^2-4x-5}\,dx = \int_0^4 \left(\frac{2/3}{x-5}+\frac{1/3}{x+1}\right)dx = \left[\tfrac{2}{3}\ln|x-5|+\tfrac{1}{3}\ln|x+1|\right]_0^4$$
$$= \tfrac{2}{3}\ln 1 + \tfrac{1}{3}\ln 5 - \tfrac{2}{3}\ln 5 - \tfrac{1}{3}\ln 1 = -\tfrac{1}{3}\ln 5$$

11. $\displaystyle\int \frac{x-1}{x^2-4x+5}\,dx = \int \frac{(x-2)+1}{(x-2)^2+1}\,dx = \int \left(\frac{u}{u^2+1}+\frac{1}{u^2+1}\right)du \quad [u=x-2,\ du=dx]$

$$= \tfrac{1}{2}\ln(u^2+1)+\tan^{-1}u + C = \tfrac{1}{2}\ln(x^2-4x+5)+\tan^{-1}(x-2)+C$$

12. $\displaystyle\int \frac{x}{x^4+x^2+1}\,dx = \int \frac{\frac{1}{2}\,du}{u^2+u+1} \quad \begin{bmatrix} u=x^2, \\ du=2x\,dx \end{bmatrix} \quad = \frac{1}{2}\int \frac{du}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}$

$$= \frac{1}{2}\int \frac{\frac{\sqrt{3}}{2}\,dv}{\frac{3}{4}(v^2+1)} \quad \begin{bmatrix} u+\frac{1}{2}=\frac{\sqrt{3}}{2}v, \\ du=\frac{\sqrt{3}}{2}\,dv \end{bmatrix} \quad = \frac{\sqrt{3}}{4}\cdot\frac{4}{3}\int\frac{dv}{v^2+1}$$
$$= \tfrac{1}{\sqrt{3}}\tan^{-1}v + C = \tfrac{1}{\sqrt{3}}\tan^{-1}\!\left(\tfrac{2}{\sqrt{3}}\left(x^2+\tfrac{1}{2}\right)\right)+C$$

13. $\int \sin^3\theta\cos^5\theta\,d\theta = \int \cos^5\theta\sin^2\theta\sin\theta\,d\theta = -\int \cos^5\theta\,(1-\cos^2\theta)(-\sin\theta)\,d\theta$

$$= -\int u^5(1-u^2)\,du \quad \begin{bmatrix} u=\cos\theta, \\ du=-\sin\theta\,d\theta \end{bmatrix}$$
$$= \int (u^7-u^5)\,du = \tfrac{1}{8}u^8 - \tfrac{1}{6}u^6 + C = \tfrac{1}{8}\cos^8\theta - \tfrac{1}{6}\cos^6\theta + C$$

Another solution:

$\int \sin^3\theta\cos^5\theta\,d\theta = \int \sin^3\theta\,(\cos^2\theta)^2\cos\theta\,d\theta = \int \sin^3\theta\,(1-\sin^2\theta)^2\cos\theta\,d\theta$

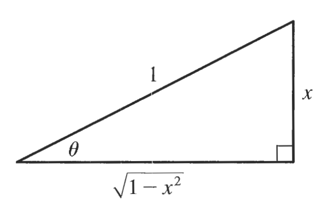
$$= \int u^3(1-u^2)^2\,du \quad \begin{bmatrix} u=\sin\theta, \\ du=\cos\theta\,d\theta \end{bmatrix} \quad = \int u^3(1-2u^2+u^4)\,du$$
$$= \int (u^3-2u^5+u^7)\,du = \tfrac{1}{4}u^4 - \tfrac{1}{3}u^6 + \tfrac{1}{8}u^8 + C = \tfrac{1}{4}\sin^4\theta - \tfrac{1}{3}\sin^6\theta + \tfrac{1}{8}\sin^8\theta + C$$

14. Let $u=1+x^2$, so that $du=2x\,dx$. Then

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{x^2}{\sqrt{1+x^2}}\,(x\,dx) = \int \frac{u-1}{u^{1/2}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int\left(u^{1/2}-u^{-1/2}\right)du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}-2u^{1/2}\right)+C$$
$$= \tfrac{1}{3}(1+x^2)^{3/2}-(1+x^2)^{1/2}+C \quad \left[\text{or } \tfrac{1}{3}(x^2-2)\sqrt{1+x^2}+C\right]$$

15. Let $x=\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Then $dx=\cos\theta\,d\theta$ and $(1-x^2)^{1/2}=\cos\theta$, so

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta\,d\theta}{(\cos\theta)^3} = \int \sec^2\theta\,d\theta = \tan\theta + C = \frac{x}{\sqrt{1-x^2}}+C.$$



16. $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \quad \begin{bmatrix} u = \sin\theta, \\ du = \cos\theta\,d\theta \end{bmatrix}$

$= \int_0^{\pi/4} \frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4}-\frac{1}{2}\right)-(0-0)\right] = \frac{\pi}{8}-\frac{1}{4}$

17. $\int x\sin^2 x\,dx \quad \begin{bmatrix} u = x, & dv = \sin^2 x\,dx, \\ du = dx & v = \int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}x - \frac{1}{2}\sin x\cos x \end{bmatrix}$

$= \frac{1}{2}x^2 - \frac{1}{2}x\sin x\cos x - \int\left(\frac{1}{2}x - \frac{1}{2}\sin x\cos x\right)dx$

$= \frac{1}{2}x^2 - \frac{1}{2}x\sin x\cos x - \frac{1}{4}x^2 + \frac{1}{4}\sin^2 x + C = \frac{1}{4}x^2 - \frac{1}{2}x\sin x\cos x + \frac{1}{4}\sin^2 x + C$

Note: $\int \sin x\cos x\,dx = \int s\,ds = \frac{1}{2}s^2 + C$ [where $s = \sin x$, $ds = \cos x\,dx$].

A slightly different method is to write $\int x\sin^2 x\,dx = \int x\cdot\frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx$. If we evaluate the second integral by parts, we arrive at the equivalent answer $\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$.

18. Let $u = e^{2t}$, $du = 2e^{2t}\,dt$. Then $\displaystyle\int \frac{e^{2t}}{1+e^{4t}}\,dt = \int \frac{\frac{1}{2}(2e^{2t})\,dt}{1+(e^{2t})^2} = \int \frac{\frac{1}{2}\,du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}(e^{2t}) + C.$

19. Let $u = e^x$. Then $\int e^{x+e^x}\,dx = \int e^{e^x}e^x\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$.

20. Since e^2 is a constant, $\int e^2\,dx = e^2x + C$.

21. Let $t = \sqrt{x}$, so that $t^2 = x$ and $2t\,dt = dx$. Then $\int \arctan\sqrt{x}\,dx = \int \arctan t\,(2t\,dt) = I$. Now use parts with $u = \arctan t$, $dv = 2t\,dt \Rightarrow du = \dfrac{1}{1+t^2}\,dt$, $v = t^2$. Thus,

$$I = t^2\arctan t - \int \frac{t^2}{1+t^2}\,dt = t^2\arctan t - \int\left(1 - \frac{1}{1+t^2}\right)dt = t^2\arctan t - t + \arctan t + C$$

$$= x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C \quad \left[\text{or } (x+1)\arctan\sqrt{x} - \sqrt{x} + C\right]$$

22. Let $u = 1 + (\ln x)^2$, so that $du = \dfrac{2\ln x}{x}\,dx$. Then

$$\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\left(2\sqrt{u}\right) + C = \sqrt{1+(\ln x)^2} + C.$$

23. Let $u = 1 + \sqrt{x}$. Then $x = (u-1)^2$, $dx = 2(u-1)\,du \Rightarrow$

$$\int_0^1 \left(1+\sqrt{x}\right)^8 dx = \int_1^2 u^8\cdot 2(u-1)\,du = 2\int_1^2 (u^9 - u^8)\,du = \left[\tfrac{1}{5}u^{10} - 2\cdot\tfrac{1}{9}u^9\right]_1^2 = \tfrac{1024}{5} - \tfrac{1024}{9} - \tfrac{1}{5} + \tfrac{2}{9} = \tfrac{4097}{45}.$$

24. Let $u = \ln(x^2-1)$, $dv = dx \Leftrightarrow du = \dfrac{2x}{x^2-1}$, $v = x$. Then

$$\int \ln(x^2-1)\,dx = x\ln(x^2-1) - \int \frac{2x^2}{x^2-1}\,dx = x\ln(x^2-1) - \int\left[2 + \frac{2}{(x-1)(x+1)}\right]dx$$

$$= x\ln(x^2-1) - \int\left[2 + \frac{1}{x-1} - \frac{1}{x+1}\right]dx = x\ln(x^2-1) - 2x - \ln|x-1| + \ln|x+1| + C$$

25. $\dfrac{3x^2-2}{x^2-2x-8} = 3 + \dfrac{6x+22}{(x-4)(x+2)} = 3 + \dfrac{A}{x-4} + \dfrac{B}{x+2} \Rightarrow 6x+22 = A(x+2)+B(x-4)$. Setting $x=4$ gives $46 = 6A$, so $A = \frac{23}{3}$. Setting $x=-2$ gives $10=-6B$, so $B = -\frac{5}{3}$. Now

$$\int \frac{3x^2-2}{x^2-2x-8}\,dx = \int \left(3 + \frac{23/3}{x-4} - \frac{5/3}{x+2}\right) dx = 3x + \tfrac{23}{3}\ln|x-4| - \tfrac{5}{3}\ln|x+2| + C.$$

26. $\displaystyle\int \frac{3x^2-2}{x^3-2x-8}\,dx = \int \frac{du}{u} \quad \begin{bmatrix} u = x^3-2x-8, \\ du = (3x^2-2)\,dx \end{bmatrix} \quad = \ln|u| + C = \ln\left|x^3-2x-8\right| + C$

27. Let $u = 1+e^x$, so that $du = e^x\,dx = (u-1)\,dx$. Then $\displaystyle\int \frac{1}{1+e^x}\,dx = \int \frac{1}{u}\cdot\frac{du}{u-1} = \int \frac{1}{u(u-1)}\,du = I$. Now

$\dfrac{1}{u(u-1)} = \dfrac{A}{u} + \dfrac{B}{u-1} \Rightarrow 1 = A(u-1) + Bu$. Set $u=1$ to get $1=B$. Set $u=0$ to get $1=-A$, so $A=-1$.

Thus, $\displaystyle I = \int \left(\frac{-1}{u} + \frac{1}{u-1}\right) du = -\ln|u| + \ln|u-1| + C = -\ln(1+e^x) + \ln e^x + C = x - \ln(1+e^x) + C.$

Another method: Multiply numerator and denominator by e^{-x} and let $u = e^{-x}+1$. This gives the answer in the form $-\ln(e^{-x}+1)+C$.

28. $\int \sin\sqrt{at}\,dt = \int \sin u \cdot \frac{2}{a}u\,du \quad [u=\sqrt{at},\, u^2 = at,\, 2u\,du = a\,dt] \quad = \frac{2}{a}\int u\sin u\,du$

$= \frac{2}{a}[-u\cos u + \sin u] + C \quad$ [integration by parts] $\quad = -\frac{2}{a}\sqrt{at}\,\cos\sqrt{at} + \frac{2}{a}\sin\sqrt{at} + C$

$= -2\sqrt{\frac{t}{a}}\,\cos\sqrt{at} + \frac{2}{a}\sin\sqrt{at} + C$

29. $\displaystyle\int_0^5 \frac{3w-1}{w+2}\,dw = \int_0^5 \left(3 - \frac{7}{w+2}\right) dw = \Big[3w - 7\ln|w+2|\Big]_0^5 = 15 - 7\ln 7 + 7\ln 2$

$= 15 + 7(\ln 2 - \ln 7) = 15 + 7\ln\frac{2}{7}$

30. $x^2-4x<0$ on $[0,4]$, so

$$\int_{-2}^{2} \left|x^2-4x\right| dx = \textstyle\int_{-2}^{0}(x^2-4x)\,dx + \int_0^2 (4x-x^2)\,dx = \left[\frac{1}{3}x^3 - 2x^2\right]_{-2}^{0} + \left[2x^2 - \frac{1}{3}x^3\right]_0^2$$

$$= 0 - \left(-\tfrac{8}{3} - 8\right) + \left(8 - \tfrac{8}{3}\right) - 0 = 16$$

31. As in Example 5,

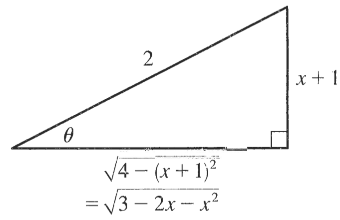
$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1}x - \sqrt{1-x^2} + C.$$

Another method: Substitute $u = \sqrt{(1+x)/(1-x)}$.

32. $\displaystyle\int \frac{\sqrt{2x-1}}{2x+3}\,dx = \int \frac{u\cdot u\,du}{u^2+4} \quad \begin{bmatrix} u=\sqrt{2x-1},\, 2x+3 = u^2+4, \\ u^2 = 2x-1,\, u\,du = dx \end{bmatrix} \quad = \int \left(1 - \frac{4}{u^2+4}\right) du$

$= u - 4\cdot\frac{1}{2}\tan^{-1}\left(\frac{1}{2}u\right) + C = \sqrt{2x-1} - 2\tan^{-1}\left(\frac{1}{2}\sqrt{2x-1}\right) + C$

33. $3 - 2x - x^2 = -(x^2 + 2x + 1) + 4 = 4 - (x+1)^2$. Let $x + 1 = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\, d\theta$ and



$$\begin{aligned}\int \sqrt{3 - 2x - x^2}\, dx &= \int \sqrt{4 - (x+1)^2}\, dx = \int \sqrt{4 - 4\sin^2\theta}\, 2\cos\theta\, d\theta \\ &= 4\int \cos^2\theta\, d\theta = 2\int (1 + \cos 2\theta)\, d\theta \\ &= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C \\ &= 2\sin^{-1}\left(\frac{x+1}{2}\right) + 2\cdot\frac{x+1}{2}\cdot\frac{\sqrt{3 - 2x - x^2}}{2} + C \\ &= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3 - 2x - x^2} + C\end{aligned}$$

34. $$\int_{\pi/4}^{\pi/2} \frac{1 + 4\cot x}{4 - \cot x}\, dx = \int_{\pi/4}^{\pi/2} \left[\frac{(1 + 4\cos x/\sin x)}{(4 - \cos x/\sin x)} \cdot \frac{\sin x}{\sin x}\right] dx = \int_{\pi/4}^{\pi/2} \frac{\sin x + 4\cos x}{4\sin x - \cos x}\, dx$$

$$= \int_{3/\sqrt{2}}^{4} \frac{1}{u}\, du \quad \left[\begin{matrix} u = 4\sin x - \cos x, \\ du = (4\cos x + \sin x)\, dx \end{matrix}\right]$$

$$= \Big[\ln|u|\Big]_{3/\sqrt{2}}^{4} = \ln 4 - \ln\frac{3}{\sqrt{2}} = \ln\frac{4}{3/\sqrt{2}} = \ln\left(\frac{4}{3}\sqrt{2}\right)$$

35. Because $f(x) = x^8 \sin x$ is the product of an even function and an odd function, it is odd. Therefore, $\int_{-1}^{1} x^8 \sin x\, dx = 0$ [by (5.5.7)(b)].

36. $\sin 4x \cos 3x = \frac{1}{2}(\sin x + \sin 7x)$ by Formula 7.2.2(a), so

$\int \sin 4x \cos 3x\, dx = \frac{1}{2}\int (\sin x + \sin 7x)\, dx = \frac{1}{2}\left[-\cos x - \frac{1}{7}\cos 7x\right] + C = -\frac{1}{2}\cos x - \frac{1}{14}\cos 7x + C.$

37. $\int_0^{\pi/4} \cos^2\theta \tan^2\theta\, d\theta = \int_0^{\pi/4} \sin^2\theta\, d\theta = \int_0^{\pi/4} \frac{1}{2}(1 - \cos 2\theta)\, d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/4} = \left(\frac{\pi}{8} - \frac{1}{4}\right) - (0 - 0) = \frac{\pi}{8} - \frac{1}{4}$

38. $$\int_0^{\pi/4} \tan^5\theta \sec^3\theta\, d\theta = \int_0^{\pi/4} (\tan^2\theta)^2 \sec^2\theta \cdot \sec\theta \tan\theta\, d\theta = \int_1^{\sqrt{2}} (u^2 - 1)^2 u^2\, du \quad \left[\begin{matrix} u = \sec\theta, \\ du = \sec\theta\tan\theta\, d\theta \end{matrix}\right]$$

$$= \int_1^{\sqrt{2}} (u^6 - 2u^4 + u^2)\, du = \left[\tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3\right]_1^{\sqrt{2}}$$

$$= \left(\tfrac{8}{7}\sqrt{2} - \tfrac{8}{5}\sqrt{2} + \tfrac{2}{3}\sqrt{2}\right) - \left(\tfrac{1}{7} - \tfrac{2}{5} + \tfrac{1}{3}\right) = \tfrac{22}{105}\sqrt{2} - \tfrac{8}{105} = \tfrac{2}{105}(11\sqrt{2} - 4)$$

39. Let $u = \sec\theta$, so that $du = \sec\theta\tan\theta\, d\theta$. Then $\displaystyle\int \frac{\sec\theta\tan\theta}{\sec^2\theta - \sec\theta}\, d\theta = \int \frac{1}{u^2 - u}\, du = \int \frac{1}{u(u-1)}\, du = I$. Now

$\displaystyle\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} \Rightarrow 1 = A(u-1) + Bu$. Set $u = 1$ to get $1 = B$. Set $u = 0$ to get $1 = -A$, so $A = -1$.

Thus, $\displaystyle I = \int \left(\frac{-1}{u} + \frac{1}{u-1}\right) du = -\ln|u| + \ln|u-1| + C = \ln|\sec\theta - 1| - \ln|\sec\theta| + C$ [or $\ln|1 - \cos\theta| + C$].

40. $4y^2 - 4y - 3 = (2y-1)^2 - 2^2$, so let $u = 2y - 1 \Rightarrow du = 2\,dy$. Thus,

$$\int \frac{dy}{\sqrt{4y^2 - 4y - 3}} = \int \frac{dy}{\sqrt{(2y-1)^2 - 2^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2 - 2^2}}$$

$$= \tfrac{1}{2}\ln\left|u + \sqrt{u^2 - 2^2}\right| \quad \text{[by Formula 20 in the table in this section]}$$

$$= \tfrac{1}{2}\ln\left|2y - 1 + \sqrt{4y^2 - 4y - 3}\right| + C$$

41. Let $u = \theta$, $dv = \tan^2\theta\,d\theta = (\sec^2\theta - 1)\,d\theta \;\Rightarrow\; du = d\theta$ and $v = \tan\theta - \theta$. So

$$\int \theta\tan^2\theta\,d\theta = \theta(\tan\theta - \theta) - \int(\tan\theta - \theta)\,d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \tfrac{1}{2}\theta^2 + C$$
$$= \theta\tan\theta - \tfrac{1}{2}\theta^2 - \ln|\sec\theta| + C$$

42. Let $u = \tan^{-1}x$, $dv = \dfrac{1}{x^2}\,dx \;\Rightarrow\; du = \dfrac{1}{1+x^2}\,dx$, $v = -\dfrac{1}{x}$. Then

$$I = \int \frac{\tan^{-1}x}{x^2}\,dx = -\frac{1}{x}\tan^{-1}x - \int\left(-\frac{1}{x(1+x^2)}\right)dx = -\frac{1}{x}\tan^{-1}x + \int\left(\frac{A}{x} + \frac{Bx+C}{1+x^2}\right)dx$$

$\dfrac{1}{x(1+x^2)} = \dfrac{A}{x} + \dfrac{Bx+C}{1+x^2} \;\Rightarrow\; 1 = A(1+x^2) + (Bx+C)x \;\Rightarrow\; 1 = (A+B)x^2 + Cx + A$, so $C = 0$, $A = 1$,

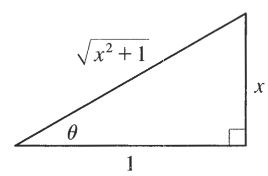
and $A + B = 0 \;\Rightarrow\; B = -1$. Thus,

$$I = -\frac{1}{x}\tan^{-1}x + \int\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx = -\frac{1}{x}\tan^{-1}x + \ln|x| - \frac{1}{2}\ln\left|1+x^2\right| + C$$
$$= -\frac{\tan^{-1}x}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$$

Or: Let $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$. Then $\displaystyle\int\frac{\tan^{-1}x}{x^2}\,dx = \int\frac{\theta}{\tan^2\theta}\sec^2\theta\,d\theta = \int\theta\csc^2\theta\,d\theta = I$. Now use parts

with $u = \theta$, $dv = \csc^2\theta\,d\theta \;\Rightarrow\; du = d\theta$, $v = -\cot\theta$. Thus,

$$I = -\theta\cot\theta - \int(-\cot\theta)\,d\theta = -\theta\cot\theta + \ln|\sin\theta| + C$$
$$= -\tan^{-1}x\cdot\frac{1}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C = -\frac{\tan^{-1}x}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$$



43. Let $u = 1 + e^x$, so that $du = e^x\,dx$. Then $\int e^x\sqrt{1+e^x}\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

Or: Let $u = \sqrt{1+e^x}$, so that $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$. Then

$\int e^x\sqrt{1+e^x}\,dx = \int u\cdot 2u\,du = \int 2u^2\,du = \frac{2}{3}u^3 + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

44. Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$, $2u\,du = e^x\,dx = (u^2-1)\,dx$, and $dx = \dfrac{2u}{u^2-1}\,du$, so

$$\int\sqrt{1+e^x}\,dx = \int u\cdot\frac{2u}{u^2-1}\,du = \int\frac{2u^2}{u^2-1}\,du = \int\left(2 + \frac{2}{u^2-1}\right)du = \int\left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right)du$$
$$= 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1+e^x} + \ln\left(\sqrt{1+e^x} - 1\right) - \ln\left(\sqrt{1+e^x} + 1\right) + C$$

45. Let $t = x^3$. Then $dt = 3x^2\,dx \;\Rightarrow\; I = \int x^5 e^{-x^3}\,dx = \frac{1}{3}\int te^{-t}\,dt$. Now integrate by parts with $u = t$, $dv = e^{-t}\,dt$:

$I = -\frac{1}{3}te^{-t} + \frac{1}{3}\int e^{-t}\,dt = -\frac{1}{3}te^{-t} - \frac{1}{3}e^{-t} + C = -\frac{1}{3}e^{-x^3}(x^3+1) + C$.

46. $\dfrac{1+\sin x}{1-\sin x}=\dfrac{1+\sin x}{1-\sin x}\cdot\dfrac{1+\sin x}{1+\sin x}=\dfrac{1+2\sin x+\sin^2 x}{1-\sin^2 x}=\dfrac{1+2\sin x+\sin^2 x}{\cos^2 x}=\dfrac{1}{\cos^2 x}+\dfrac{2\sin x}{\cos^2 x}+\dfrac{\sin^2 x}{\cos^2 x}$

$=\sec^2 x+2\sec x\,\tan x+\tan^2 x=\sec^2 x+2\sec x\,\tan x+\sec^2 x-1=2\sec^2 x+2\sec x\,\tan x-1$

Thus, $\displaystyle\int\frac{1+\sin x}{1-\sin x}\,dx=\int(2\sec^2 x+2\sec x\,\tan x-1)\,dx=2\tan x+2\sec x-x+C$

47. Let $u=x-1$, so that $du=dx$. Then

$\int x^3(x-1)^{-4}\,dx=\int(u+1)^3u^{-4}\,du=\int(u^3+3u^2+3u+1)u^{-4}\,du=\int(u^{-1}+3u^{-2}+3u^{-3}+u^{-4})\,du$

$=\ln|u|-3u^{-1}-\frac{3}{2}u^{-2}-\frac{1}{3}u^{-3}+C=\ln|x-1|-3(x-1)^{-1}-\frac{3}{2}(x-1)^{-2}-\frac{1}{3}(x-1)^{-3}+C$

48. Let $u=x^2$. Then $du=2x\,dx\;\Rightarrow\;\displaystyle\int\frac{x\,dx}{x^4-a^4}=\int\frac{\frac{1}{2}\,du}{u^2-(a^2)^2}=\frac{1}{4a^2}\ln\left|\frac{u-a^2}{u+a^2}\right|+C=\frac{1}{4a^2}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right|+C.$

49. Let $u=\sqrt{4x+1}\;\Rightarrow\;u^2=4x+1\;\Rightarrow\;2u\,du=4\,dx\;\Rightarrow\;dx=\frac{1}{2}u\,du$. So

$$\int\frac{1}{x\sqrt{4x+1}}\,dx=\int\frac{\frac{1}{2}u\,du}{\frac{1}{4}(u^2-1)\,u}=2\int\frac{du}{u^2-1}=2\left(\tfrac{1}{2}\right)\ln\left|\frac{u-1}{u+1}\right|+C\qquad\text{[by Formula 19]}$$

$$=\ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right|+C$$

50. As in Exercise 49, let $u=\sqrt{4x+1}$. Then $\displaystyle\int\frac{dx}{x^2\sqrt{4x+1}}=\int\frac{\frac{1}{2}u\,du}{\left[\frac{1}{4}(u^2-1)\right]^2u}=8\int\frac{du}{(u^2-1)^2}.$

Now $\dfrac{1}{(u^2-1)^2}=\dfrac{1}{(u+1)^2(u-1)^2}=\dfrac{A}{u+1}+\dfrac{B}{(u+1)^2}+\dfrac{C}{u-1}+\dfrac{D}{(u-1)^2}\;\Rightarrow$

$1=A(u+1)(u-1)^2+B(u-1)^2+C(u-1)(u+1)^2+D(u+1)^2$. $u=1\;\Rightarrow\;D=\frac{1}{4}$, $u=-1\;\Rightarrow\;B=\frac{1}{4}$.

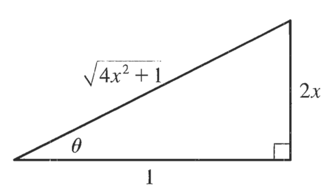
Equating coefficients of u^3 gives $A+C=0$, and equating coefficients of 1 gives $1=A+B-C+D\;\Rightarrow$

$1=A+\frac{1}{4}-C+\frac{1}{4}\;\Rightarrow\;\frac{1}{2}=A-C$. So $A=\frac{1}{4}$ and $C=-\frac{1}{4}$. Therefore,

$$\begin{aligned}\int\frac{dx}{x^2\sqrt{4x+1}}&=8\int\left[\frac{1/4}{u+1}+\frac{1/4}{(u+1)^2}+\frac{-1/4}{u-1}+\frac{1/4}{(u-1)^2}\right]du\\&=\int\left[\frac{2}{u+1}+2(u+1)^{-2}-\frac{2}{u-1}+2(u-1)^{-2}\right]du\\&=2\ln|u+1|-\frac{2}{u+1}-2\ln|u-1|-\frac{2}{u-1}+C\\&=2\ln\left(\sqrt{4x+1}+1\right)-\frac{2}{\sqrt{4x+1}+1}-2\ln\left|\sqrt{4x+1}-1\right|-\frac{2}{\sqrt{4x+1}-1}+C\end{aligned}$$

51. Let $2x=\tan\theta\;\Rightarrow\;x=\frac{1}{2}\tan\theta$, $dx=\frac{1}{2}\sec^2\theta\,d\theta$, $\sqrt{4x^2+1}=\sec\theta$, so

$$\begin{aligned}\int\frac{dx}{x\sqrt{4x^2+1}}&=\int\frac{\frac{1}{2}\sec^2\theta\,d\theta}{\frac{1}{2}\tan\theta\,\sec\theta}=\int\frac{\sec\theta}{\tan\theta}\,d\theta=\int\csc\theta\,d\theta\\&=-\ln|\csc\theta+\cot\theta|+C\qquad[\text{or }\ln|\csc\theta-\cot\theta|+C]\\&=-\ln\left|\frac{\sqrt{4x^2+1}}{2x}+\frac{1}{2x}\right|+C\qquad\left[\text{or }\ln\left|\frac{\sqrt{4x^2+1}}{2x}-\frac{1}{2x}\right|+C\right]\end{aligned}$$



52. Let $u = x^2$. Then $du = 2x\,dx \Rightarrow$

$$\int \frac{dx}{x(x^4+1)} = \int \frac{x\,dx}{x^2(x^4+1)} = \frac{1}{2}\int \frac{du}{u(u^2+1)} = \frac{1}{2}\int \left[\frac{1}{u} - \frac{u}{u^2+1}\right] du = \tfrac{1}{2}\ln|u| - \tfrac{1}{4}\ln(u^2+1) + C$$

$$= \tfrac{1}{2}\ln(x^2) - \tfrac{1}{4}\ln(x^4+1) + C = \tfrac{1}{4}\left[\ln(x^4) - \ln(x^4+1)\right] + C = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C$$

Or: Write $I = \displaystyle\int \frac{x^3\,dx}{x^4(x^4+1)}$ and let $u = x^4$.

53. $\displaystyle\int x^2 \sinh(mx)dx = \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\int x\cosh(mx)\,dx$ $\quad\left[\begin{array}{ll} u = x^2, & dv = \sinh(mx)\,dx, \\ du = 2x\,dx & v = \frac{1}{m}\cosh(mx) \end{array}\right]$

$$= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\left(\tfrac{1}{m}x\sinh(mx) - \frac{1}{m}\textstyle\int \sinh(mx)\,dx\right) \quad \left[\begin{array}{ll} U = x, & dV = \cosh(mx)\,dx, \\ dU = dx & V = \frac{1}{m}\sinh(mx) \end{array}\right]$$

$$= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m^2}x\sinh(mx) + \frac{2}{m^3}\cosh(mx) + C$$

54. $\int (x+\sin x)^2\,dx = \int \left(x^2 + 2x\sin x + \sin^2 x\right)dx = \frac{1}{3}x^3 + 2(\sin x - x\cos x) + \frac{1}{2}(x - \sin x\cos x) + C$

$$= \tfrac{1}{3}x^3 + \tfrac{1}{2}x + 2\sin x - \tfrac{1}{2}\sin x\cos x - 2x\cos x + C$$

55. Let $u = \sqrt{x}$, so that $x = u^2$ and $dx = 2u\,du$. Then $\displaystyle\int \frac{dx}{x + x\sqrt{x}} = \int \frac{2u\,du}{u^2 + u^2\cdot u} = \int \frac{2}{u(1+u)}\,du = I.$

Now $\displaystyle\frac{2}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u} \Rightarrow 2 = A(1+u) + Bu$. Set $u = -1$ to get $2 = -B$, so $B = -2$. Set $u = 0$ to get $2 = A$.

Thus, $\displaystyle I = \int\left(\frac{2}{u} - \frac{2}{1+u}\right)du = 2\ln|u| - 2\ln|1+u| + C = 2\ln\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C.$

56. Let $u = \sqrt{x}$, so that $x = u^2$ and $dx = 2u\,du$. Then

$$\int \frac{dx}{\sqrt{x} + x\sqrt{x}} = \int \frac{2u\,du}{u + u^2\cdot u} = \int \frac{2}{1+u^2}\,du = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C.$$

57. Let $u = \sqrt[3]{x+c}$. Then $x = u^3 - c \Rightarrow$

$$\int x\sqrt[3]{x+c}\,dx = \int (u^3 - c)u\cdot 3u^2\,du = 3\int (u^6 - cu^3)\,du = \tfrac{3}{7}u^7 - \tfrac{3}{4}cu^4 + C = \tfrac{3}{7}(x+c)^{7/3} - \tfrac{3}{4}c(x+c)^{4/3} + C$$

58. Let $t = \sqrt{x^2-1}$. Then $dt = \left(x/\sqrt{x^2-1}\right)dx$, $x^2 - 1 = t^2$, $x = \sqrt{t^2+1}$, so

$\displaystyle I = \int \frac{x\ln x}{\sqrt{x^2-1}}\,dx = \int \ln\sqrt{t^2+1}\,dt = \frac{1}{2}\int \ln(t^2+1)\,dt$. Now use parts with $u = \ln(t^2+1)$, $dv = dt$:

$$I = \tfrac{1}{2}t\ln(t^2+1) - \int \frac{t^2}{t^2+1}\,dt = \tfrac{1}{2}t\ln(t^2+1) - \int\left[1 - \frac{1}{t^2+1}\right]dt$$

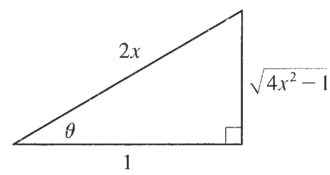
$$= \tfrac{1}{2}t\ln(t^2+1) - t + \tan^{-1}t + C = \sqrt{x^2-1}\,\ln x - \sqrt{x^2-1} + \tan^{-1}\sqrt{x^2-1} + C$$

Another method: First integrate by parts with $u = \ln x$, $dv = \left(x/\sqrt{x^2-1}\right)dx$ and then use substitution $\left(x = \sec\theta \text{ or } u = \sqrt{x^2-1}\right)$.

59. Let $u = \sin x$, so that $du = \cos x\, dx$. Then

$$\int \cos x \cos^3(\sin x)\, dx = \int \cos^3 u\, du = \int \cos^2 u \cos u\, du = \int (1 - \sin^2 u) \cos u\, du$$
$$= \int (\cos u - \sin^2 u \cos u)\, du = \sin u - \tfrac{1}{3}\sin^3 u + C = \sin(\sin x) - \tfrac{1}{3}\sin^3(\sin x) + C$$

60. Let $2x = \sec\theta$, so that $2\, dx = \sec\theta \tan\theta\, d\theta$. Then



$$\int \frac{dx}{x^2\sqrt{4x^2-1}} = \int \frac{\frac{1}{2}\sec\theta \tan\theta\, d\theta}{\frac{1}{4}\sec^2\theta\sqrt{\sec^2\theta - 1}} = \int \frac{2\tan\theta\, d\theta}{\sec\theta \tan\theta}$$
$$= 2\int \cos\theta\, d\theta = 2\sin\theta + C$$
$$= 2 \cdot \frac{\sqrt{4x^2-1}}{2x} + C = \frac{\sqrt{4x^2-1}}{x} + C$$

61. Let $y = \sqrt{x}$ so that $dy = \dfrac{1}{2\sqrt{x}}\, dx \quad\Rightarrow\quad dx = 2\sqrt{x}\, dy = 2y\, dy$. Then

$$\int \sqrt{x}\, e^{\sqrt{x}}\, dx = \int ye^y(2y\, dy) = \int 2y^2 e^y\, dy \qquad \begin{bmatrix} u = 2y^2, & dv = e^y\, dy, \\ du = 4y\, dy & v = e^y \end{bmatrix}$$
$$= 2y^2e^y - \int 4ye^y\, dy \qquad \begin{bmatrix} U = 4y, & dV = e^y\, dy, \\ dU = 4\, dy & V = e^y \end{bmatrix}$$
$$= 2y^2e^y - \left(4ye^y - \int 4e^y\, dy\right) = 2y^2e^y - 4ye^y + 4e^y + C$$
$$= 2(y^2 - 2y + 2)e^y + C = 2\left(x - 2\sqrt{x} + 2\right)e^{\sqrt{x}} + C$$

62. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2\, du \quad\Rightarrow$

$$\int \frac{dx}{x + \sqrt[3]{x}} = \int \frac{3u^2\, du}{u^3 + u} = \frac{3}{2}\int \frac{2u\, du}{u^2+1} = \tfrac{3}{2}\ln(u^2+1) + C = \tfrac{3}{2}\ln(x^{2/3}+1) + C.$$

63. Let $u = \cos^2 x$, so that $du = 2\cos x\,(-\sin x)\, dx$. Then

$$\int \frac{\sin 2x}{1 + \cos^4 x}\, dx = \int \frac{2\sin x \cos x}{1 + (\cos^2 x)^2}\, dx = \int \frac{1}{1+u^2}(-du) = -\tan^{-1} u + C = -\tan^{-1}(\cos^2 x) + C.$$

64. Let $u = \tan x$. Then

$$\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)\, dx}{\sin x \cos x} = \int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\tan x}\sec^2 x\, dx = \int_1^{\sqrt{3}} \frac{\ln u}{u}\, du = \left[\tfrac{1}{2}(\ln u)^2\right]_1^{\sqrt{3}} = \tfrac{1}{2}\left(\ln\sqrt{3}\right)^2 = \tfrac{1}{8}(\ln 3)^2.$$

65. $$\int \frac{dx}{\sqrt{x+1} + \sqrt{x}} = \int \left(\frac{1}{\sqrt{x+1}+\sqrt{x}} \cdot \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\right) dx = \int \left(\sqrt{x+1} - \sqrt{x}\right) dx$$
$$= \tfrac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C$$

66. $$\int \frac{u^3+1}{u^3-u^2}\, du = \int \left[1 + \frac{u^2+1}{(u-1)u^2}\right] du = u + \int \left[\frac{2}{u-1} - \frac{1}{u} - \frac{1}{u^2}\right] du = u + 2\ln|u-1| - \ln|u| + \frac{1}{u} + C.$$ Thus,

$$\int_2^3 \frac{u^3+1}{u^3-u^2}\, du = \left[u + 2\ln(u-1) - \ln u + \frac{1}{u}\right]_2^3 = \left(3 + 2\ln 2 - \ln 3 + \tfrac{1}{3}\right) - \left(2 + 2\ln 1 - \ln 2 + \tfrac{1}{2}\right)$$
$$= 1 + 3\ln 2 - \ln 3 - \tfrac{1}{6} = \tfrac{5}{6} + \ln\tfrac{8}{3}$$

67. Let $x = \tan\theta$, so that $dx = \sec^2\theta\, d\theta$, $x = \sqrt{3} \;\Rightarrow\; \theta = \frac{\pi}{3}$, and $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$. Then

$$\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\,dx = \int_{\pi/4}^{\pi/3} \frac{\sec\theta}{\tan^2\theta}\sec^2\theta\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\sec\theta\,(\tan^2\theta+1)}{\tan^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3}\left(\frac{\sec\theta\,\tan^2\theta}{\tan^2\theta} + \frac{\sec\theta}{\tan^2\theta}\right)d\theta$$

$$= \int_{\pi/4}^{\pi/3} (\sec\theta + \csc\theta\,\cot\theta)\,d\theta = \Big[\ln|\sec\theta+\tan\theta| - \csc\theta\Big]_{\pi/4}^{\pi/3}$$

$$= \left(\ln\left|2+\sqrt{3}\right| - \tfrac{2}{\sqrt{3}}\right) - \left(\ln\left|\sqrt{2}+1\right| - \sqrt{2}\right) = \sqrt{2} - \tfrac{2}{\sqrt{3}} + \ln\left(2+\sqrt{3}\right) - \ln\left(1+\sqrt{2}\right)$$

68. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \;\Rightarrow$

$$\int \frac{dx}{1+2e^x - e^{-x}} = \int \frac{du/u}{1+2u-1/u} = \int \frac{du}{2u^2+u-1} = \int \left[\frac{2/3}{2u-1} - \frac{1/3}{u+1}\right]du$$

$$= \tfrac{1}{3}\ln|2u-1| - \tfrac{1}{3}\ln|u+1| + C = \tfrac{1}{3}\ln|(2e^x-1)/(e^x+1)| + C$$

69. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \;\Rightarrow$

$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{u^2}{1+u}\frac{du}{u} = \int \frac{u}{1+u}\,du = \int \left(1 - \frac{1}{1+u}\right)du = u - \ln|1+u| + C = e^x - \ln(1+e^x) + C.$$

70. Use parts with $u = \ln(x+1)$, $dv = dx/x^2$:

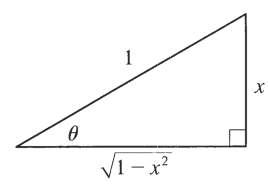
$$\int \frac{\ln(x+1)}{x^2}\,dx = -\frac{1}{x}\ln(x+1) + \int \frac{dx}{x(x+1)} = -\frac{1}{x}\ln(x+1) + \int\left[\frac{1}{x} - \frac{1}{x+1}\right]dx$$

$$= -\frac{1}{x}\ln(x+1) + \ln|x| - \ln(x+1) + C = -\left(1+\frac{1}{x}\right)\ln(x+1) + \ln|x| + C$$

71. Let $\theta = \arcsin x$, so that $d\theta = \dfrac{1}{\sqrt{1-x^2}}\,dx$ and $x = \sin\theta$. Then

$$\int \frac{x+\arcsin x}{\sqrt{1-x^2}}\,dx = \int (\sin\theta + \theta)\,d\theta = -\cos\theta + \tfrac{1}{2}\theta^2 + C$$

$$= -\sqrt{1-x^2} + \tfrac{1}{2}(\arcsin x)^2 + C$$



72. $\displaystyle\int \frac{4^x+10^x}{2^x}\,dx = \int\left(\frac{4^x}{2^x} + \frac{10^x}{2^x}\right)dx = \int (2^x+5^x)\,dx = \frac{2^x}{\ln 2} + \frac{5^x}{\ln 5} + C$

73. $\displaystyle\frac{1}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4} \;\Rightarrow\; 1 = A(x^2+4) + (Bx+C)(x-2) = (A+B)x^2 + (C-2B)x + (4A-2C).$

So $0 = A+B = C-2B$, $1 = 4A-2C$. Setting $x = 2$ gives $A = \frac{1}{8} \;\Rightarrow\; B = -\frac{1}{8}$ and $C = -\frac{1}{4}$. So

$$\int \frac{1}{(x-2)(x^2+4)}\,dx = \int\left(\frac{\frac{1}{8}}{x-2} + \frac{-\frac{1}{8}x - \frac{1}{4}}{x^2+4}\right)dx = \frac{1}{8}\int\frac{dx}{x-2} - \frac{1}{16}\int\frac{2x\,dx}{x^2+4} - \frac{1}{4}\int\frac{dx}{x^2+4}$$

$$= \tfrac{1}{8}\ln|x-2| - \tfrac{1}{16}\ln(x^2+4) - \tfrac{1}{8}\tan^{-1}(x/2) + C$$

74. Let $u = 2 + \sqrt{x}$, so that $du = \dfrac{1}{2\sqrt{x}}\,dx$. Then

$$\int \frac{dx}{\sqrt{x}\,(2+\sqrt{x})^4} = \int \frac{2\,du}{u^4} = 2\int u^{-4}\,du = -\frac{2}{3}u^{-3} + C = -\frac{2}{3\,(2+\sqrt{x})^3} + C.$$

75. Let $y = \sqrt{1+e^x}$, so that $y^2 = 1 + e^x$, $2y\,dy = e^x\,dx$, $e^x = y^2 - 1$, and $x = \ln(y^2-1)$. Then

$$\begin{aligned}\int \frac{xe^x}{\sqrt{1+e^x}}\,dx &= \int \frac{\ln(y^2-1)}{y}(2y\,dy) = 2\int [\ln(y+1) + \ln(y-1)]\,dy \\ &= 2[(y+1)\ln(y+1) - (y+1) + (y-1)\ln(y-1) - (y-1)] + C \qquad \text{[by Example 7.1.2]} \\ &= 2[y\ln(y+1) + \ln(y+1) - y - 1 + y\ln(y-1) - \ln(y-1) - y + 1] + C \\ &= 2[y(\ln(y+1) + \ln(y-1)) + \ln(y+1) - \ln(y-1) - 2y] + C \\ &= 2\left[y\ln(y^2-1) + \ln\frac{y+1}{y-1} - 2y\right] + C = 2\left[\sqrt{1+e^x}\,\ln(e^x) + \ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} - 2\sqrt{1+e^x}\right] + C \\ &= 2x\sqrt{1+e^x} + 2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} - 4\sqrt{1+e^x} + C = 2(x-2)\sqrt{1+e^x} + 2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} + C\end{aligned}$$

76.
$$\begin{aligned}\int (x^2 - bx)\sin 2x\,dx &= -\tfrac{1}{2}(x^2-bx)\cos 2x + \tfrac{1}{2}\int (2x-b)\cos 2x\,dx \\ &\qquad \left[u = x^2 - bx,\ dv = \sin 2x\,dx,\ du = (2x-b)\,dx,\ v = -\tfrac{1}{2}\cos 2x\right] \\ &= -\tfrac{1}{2}(x^2-bx)\cos 2x + \tfrac{1}{2}\left[\tfrac{1}{2}(2x-b)\sin 2x - \int \sin 2x\,dx\right] \\ &\qquad \left[U = 2x - b,\ dV = \cos 2x\,dx,\ dU = 2\,dx,\ V = \tfrac{1}{2}\sin 2x\right] \\ &= -\tfrac{1}{2}(x^2-bx)\cos 2x + \tfrac{1}{4}(2x-b)\sin 2x + \tfrac{1}{4}\cos 2x + C\end{aligned}$$

77. Let $u = x^{3/2}$ so that $u^2 = x^3$ and $du = \frac{3}{2}x^{1/2}\,dx \;\Rightarrow\; \sqrt{x}\,dx = \frac{2}{3}\,du$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{\frac{2}{3}}{1+u^2}\,du = \frac{2}{3}\tan^{-1}u + C = \tfrac{2}{3}\tan^{-1}(x^{3/2}) + C.$$

78.
$$\begin{aligned}\int \frac{\sec x\,\cos 2x}{\sin x + \sec x}\,dx &= \int \frac{\sec x\,\cos 2x}{\sin x + \sec x}\cdot\frac{2\cos x}{2\cos x}\,dx = \int \frac{2\cos 2x}{2\sin x\cos x + 2}\,dx \\ &= \int \frac{2\cos 2x}{\sin 2x + 2}\,dx = \int \frac{1}{u}\,du \qquad \left[\begin{matrix} u = \sin 2x + 2, \\ du = 2\cos 2x\,dx \end{matrix}\right] \\ &= \ln|u| + C = \ln|\sin 2x + 2| + C = \ln(\sin 2x + 2) + C\end{aligned}$$

79. Let $u = x$, $dv = \sin^2 x\cos x\,dx \;\Rightarrow\; du = dx$, $v = \frac{1}{3}\sin^3 x$. Then

$$\begin{aligned}\int x\sin^2 x\,\cos x\,dx &= \tfrac{1}{3}x\sin^3 x - \int \tfrac{1}{3}\sin^3 x\,dx = \tfrac{1}{3}x\sin^3 x - \tfrac{1}{3}\int (1-\cos^2 x)\sin x\,dx \\ &= \frac{1}{3}x\sin^3 x + \frac{1}{3}\int (1-y^2)\,dy \qquad \left[\begin{matrix} u = \cos x, \\ du = -\sin x\,dx \end{matrix}\right] \\ &= \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}y - \tfrac{1}{9}y^3 + C = \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}\cos x - \tfrac{1}{9}\cos^3 x + C\end{aligned}$$

80. $\displaystyle\int \frac{\sin x\,\cos x}{\sin^4 x+\cos^4 x}\,dx=\int \frac{\sin x\,\cos x}{(\sin^2 x)^2+(\cos^2 x)^2}\,dx=\int \frac{\sin x\,\cos x}{(\sin^2 x)^2+(1-\sin^2 x)^2}\,dx$

$$=\int \frac{1}{u^2+(1-u)^2}\left(\frac{1}{2}\,du\right)\qquad\left[\begin{array}{l}u=\sin^2 x,\\ du=2\sin x\,\cos x\,dx\end{array}\right]$$

$$=\int \frac{1}{4u^2-4u+2}\,du=\int \frac{1}{(4u^2-4u+1)+1}\,du$$

$$=\int \frac{1}{(2u-1)^2+1}\,du=\frac{1}{2}\int \frac{1}{y^2+1}\,dy\qquad\left[\begin{array}{l}y=2u-1,\\ dy=2\,du\end{array}\right]$$

$$=\tfrac{1}{2}\tan^{-1}y+C=\tfrac{1}{2}\tan^{-1}(2u-1)+C=\tfrac{1}{2}\tan^{-1}(2\sin^2 x-1)+C$$

Another solution: $\displaystyle\int \frac{\sin x\,\cos x}{\sin^4 x+\cos^4 x}\,dx=\int \frac{(\sin x\,\cos x)/\cos^4 x}{(\sin^4 x+\cos^4 x)/\cos^4 x}\,dx=\int \frac{\tan x\,\sec^2 x}{\tan^4 x+1}\,dx$

$$=\int \frac{1}{u^2+1}\left(\frac{1}{2}\,du\right)\qquad\left[\begin{array}{l}u=\tan^2 x,\\ du=2\tan x\,\sec^2 x\,dx\end{array}\right]$$

$$=\tfrac{1}{2}\tan^{-1}u+C=\tfrac{1}{2}\tan^{-1}(\tan^2 x)+C$$

81. The function $y=2xe^{x^2}$ *does* have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\int(2x^2+1)e^{x^2}\,dx=\int 2x^2e^{x^2}\,dx+\int e^{x^2}\,dx=\int x\left(2xe^{x^2}\right)dx+\int e^{x^2}\,dx$$

$$=xe^{x^2}-\int e^{x^2}\,dx+\int e^{x^2}\,dx\qquad\left[\begin{array}{ll}u=x, & dv=2xe^{x^2}\,dx,\\ du=dx & v=e^{x^2}\end{array}\right]\qquad=xe^{x^2}+C$$