7.8 Improper Integrals

1. (a) Since $\int_1^\infty x^4 e^{-x^4}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(b) Since $y = \sec x$ has an infinite discontinuity at $x = \frac{\pi}{2}$, $\int_0^{\pi/2} \sec x\,dx$ is a Type II improper integral.

(c) Since $y = \dfrac{x}{(x-2)(x-3)}$ has an infinite discontinuity at $x = 2$, $\displaystyle\int_0^2 \frac{x}{x^2 - 5x + 6}\,dx$ is a Type II improper integral.

(d) Since $\displaystyle\int_{-\infty}^0 \frac{1}{x^2+5}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

2. (a) Since $y = \dfrac{1}{2x-1}$ is defined and continuous on $[1, 2]$, $\displaystyle\int_1^2 \frac{1}{2x-1}\,dx$ is proper.

(b) Since $y = \dfrac{1}{2x-1}$ has an infinite discontinuity at $x = \frac{1}{2}$, $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

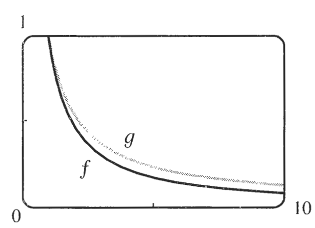
(c) Since $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

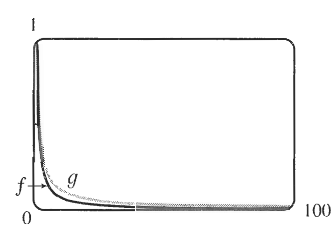
(d) Since $y = \ln(x-1)$ has an infinite discontinuity at $x = 1$, $\int_1^2 \ln(x-1)dx$ is a Type II improper integral.

3. The area under the graph of $y = 1/x^3 = x^{-3}$ between $x = 1$ and $x = t$ is

$A(t) = \int_1^t x^{-3}\,dx = \left[-\frac{1}{2}x^{-2}\right]_1^t = -\frac{1}{2}t^{-2} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 1/(2t^2)$. So the area for $1 \le x \le 10$ is $A(10) = 0.5 - 0.005 = 0.495$, the area for $1 \le x \le 100$ is $A(100) = 0.5 - 0.00005 = 0.49995$, and the area for $1 \le x \le 1000$ is $A(1000) = 0.5 - 0.0000005 = 0.4999995$. The total area under the curve for $x \ge 1$ is $\lim\limits_{t\to\infty} A(t) = \lim\limits_{t\to\infty}\left[\frac{1}{2} - 1/(2t^2)\right] = \frac{1}{2}$.

4. (a)





(b) The area under the graph of f from $x = 1$ to $x = t$ is

$$F(t) = \int_1^t f(x)\,dx = \int_1^t x^{-1.1}\,dx = \left[-\frac{1}{0.1}x^{-0.1}\right]_1^t$$
$$= -10(t^{-0.1} - 1) = 10(1 - t^{-0.1})$$

and the area under the graph of g is

$$G(t) = \int_1^t g(x)\,dx = \int_1^t x^{-0.9}\,dx = \left[\frac{1}{0.1}x^{0.1}\right]_1^t = 10(t^{0.1} - 1).$$

t	$F(t)$	$G(t)$
10	2.06	2.59
100	3.69	5.85
10^4	6.02	15.12
10^6	7.49	29.81
10^{10}	9	90
10^{20}	9.9	990

(c) The total area under the graph of f is $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} 10(1 - t^{-0.1}) = 10$.

The total area under the graph of g does not exist, since $\lim_{t\to\infty} G(t) = \lim_{t\to\infty} 10(t^{0.1} - 1) = \infty$.

5. $I = \displaystyle\int_1^\infty \frac{1}{(3x+1)^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{(3x+1)^2}\,dx$. Now

$\displaystyle\int \frac{1}{(3x+1)^2}\,dx = \frac{1}{3}\int \frac{1}{u^2}\,du \quad [u = 3x+1,\, du = 3\,dx] \quad = -\frac{1}{3u} + C = -\frac{1}{3(3x+1)} + C,$

so $I = \displaystyle\lim_{t\to\infty}\left[-\frac{1}{3(3x+1)}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{3(3t+1)} + \frac{1}{12}\right] = 0 + \frac{1}{12} = \frac{1}{12}$. Convergent

6. $\displaystyle\int_{-\infty}^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\left[\tfrac{1}{2}\ln|2x-5|\right]_t^0 = \lim_{t\to\infty}\left[\tfrac{1}{2}\ln 5 - \tfrac{1}{2}\ln|2t-5|\right] = -\infty$.

Divergent

7. $\displaystyle\int_{-\infty}^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\int_t^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\left[-2\sqrt{2-w}\right]_t^{-1} \quad [u = 2-w,\, du = -dw]$

$\displaystyle = \lim_{t\to-\infty}\left[-2\sqrt{3} + 2\sqrt{2-t}\right] = \infty$. Divergent

8. $\displaystyle\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\frac{1}{2}\left[\frac{-1}{x^2+2}\right]_0^t = \frac{1}{2}\lim_{t\to\infty}\left(\frac{-1}{t^2+2} + \frac{1}{2}\right)$

$= \tfrac{1}{2}\left(0 + \tfrac{1}{2}\right) = \tfrac{1}{4}$. Convergent

9. $\int_4^\infty e^{-y/2}\,dy = \lim_{t\to\infty}\int_4^t e^{-y/2}\,dy = \lim_{t\to\infty}\left[-2e^{-y/2}\right]_4^t = \lim_{t\to\infty}(-2e^{-t/2} + 2e^{-2}) = 0 + 2e^{-2} = 2e^{-2}$.

Convergent

10. $\int_{-\infty}^{-1} e^{-2t}\,dt = \lim_{x\to-\infty}\int_x^{-1} e^{-2t}\,dt = \lim_{x\to-\infty}\left[-\tfrac{1}{2}e^{-2t}\right]_x^{-1} = \lim_{x\to-\infty}\left[-\tfrac{1}{2}e^2 + \tfrac{1}{2}e^{-2x}\right] = \infty$. Divergent

11. $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{1+x^2} = \int_{-\infty}^0 \frac{x\,dx}{1+x^2} + \int_0^\infty \frac{x\,dx}{1+x^2}$ and

$\displaystyle\int_{-\infty}^0 \frac{x\,dx}{1+x^2} = \lim_{t\to-\infty}\left[\tfrac{1}{2}\ln(1+x^2)\right]_t^0 = \lim_{t\to-\infty}\left[0 - \tfrac{1}{2}\ln(1+t^2)\right] = -\infty$. Divergent

12. $I = \int_{-\infty}^\infty (2 - v^4)\,dv = I_1 + I_2 = \int_{-\infty}^0 (2 - v^4)\,dv + \int_0^\infty (2 - v^4)\,dv$, but

$I_1 = \lim_{t\to-\infty}\left[2v - \tfrac{1}{5}v^5\right]_t^0 = \lim_{t\to-\infty}\left(-2t + \tfrac{1}{5}t^5\right) = -\infty$. Since I_1 is divergent, I is divergent, and there is no need to

evaluate I_2. Divergent

13. $\int_{-\infty}^\infty xe^{-x^2}\,dx = \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx$.

$\int_{-\infty}^0 xe^{-x^2}\,dx = \lim_{t\to-\infty}\left(-\tfrac{1}{2}\right)\left[e^{-x^2}\right]_t^0 = \lim_{t\to-\infty}\left(-\tfrac{1}{2}\right)\left(1 - e^{-t^2}\right) = -\tfrac{1}{2}\cdot 1 = -\tfrac{1}{2}$, and

$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\left(-\tfrac{1}{2}\right)\left[e^{-x^2}\right]_0^t = \lim_{t\to\infty}\left(-\tfrac{1}{2}\right)\left(e^{-t^2} - 1\right) = -\tfrac{1}{2}\cdot(-1) = \tfrac{1}{2}$.

Therefore, $\int_{-\infty}^\infty xe^{-x^2}\,dx = -\tfrac{1}{2} + \tfrac{1}{2} = 0$. Convergent

14. $\displaystyle\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_1^{\sqrt{t}} e^{-u}(2\,du)$ $\left[\begin{array}{c} u=\sqrt{x}, \\ du = dx/\,(2\sqrt{x}\,) \end{array}\right]$

$\displaystyle = 2\lim_{t\to\infty}\left[-e^{-u}\right]_1^{\sqrt{t}} = 2\lim_{t\to\infty}\left(-e^{-\sqrt{t}}+e^{-1}\right) = 2(0+e^{-1}) = 2e^{-1}.$ Convergent

15. $\int_{2\pi}^{\infty}\sin\theta\,d\theta = \lim_{t\to\infty}\int_{2\pi}^{t}\sin\theta\,d\theta = \lim_{t\to\infty}\left[-\cos\theta\right]_{2\pi}^{t} = \lim_{t\to\infty}(-\cos t+1)$. This limit does not exist, so the integral is divergent. Divergent

16. $I = \int_{-\infty}^{\infty}\cos\pi t\,dt = I_1 + I_2 = \int_{-\infty}^{0}\cos\pi t\,dt + \int_0^\infty \cos\pi t\,dt$, but $I_1 = \displaystyle\lim_{s\to-\infty}\left[\frac{1}{\pi}\sin\pi t\right]_s^0 = \lim_{s\to-\infty}\left(-\frac{1}{\pi}\sin\pi t\right)$ and this limit does not exist. Since I_1 is divergent, I is divergent, and there is no need to evaluate I_2. Divergent

17. $\displaystyle\int_1^\infty \frac{x+1}{x^2+2x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(2x+2)}{x^2+2x}\,dx = \tfrac{1}{2}\lim_{t\to\infty}\left[\ln(x^2+2x)\right]_1^t = \tfrac{1}{2}\lim_{t\to\infty}\left[\ln(t^2+2t)-\ln 3\right] = \infty.$

Divergent

18. $\displaystyle\int_0^\infty \frac{dz}{z^2+3z+2} = \lim_{t\to\infty}\int_0^t\left[\frac{1}{z+1}-\frac{1}{z+2}\right]dz = \lim_{t\to\infty}\left[\ln\left(\frac{z+1}{z+2}\right)\right]_0^t$

$\displaystyle = \lim_{t\to\infty}\left[\ln\left(\frac{t+1}{t+2}\right)-\ln\left(\frac{1}{2}\right)\right] = \ln 1 + \ln 2 = \ln 2.$ Convergent

19. $\displaystyle\int_0^\infty se^{-5s}\,ds = \lim_{t\to\infty}\int_0^t se^{-5s}\,ds = \lim_{t\to\infty}\left[-\tfrac{1}{5}se^{-5s}-\tfrac{1}{25}e^{-5s}\right]$ [by integration by parts with $u=s$]

$\displaystyle = \lim_{t\to\infty}\left(-\tfrac{1}{5}te^{-5t}-\tfrac{1}{25}e^{-5t}+\tfrac{1}{25}\right) = 0-0+\tfrac{1}{25}$ [by l'Hospital's Rule]

$= \frac{1}{25}.$ Convergent

20. $\displaystyle\int_{-\infty}^{6} re^{r/3}\,dr = \lim_{t\to-\infty}\int_t^6 re^{r/3}\,dr = \lim_{t\to-\infty}\left[3re^{r/3}-9e^{r/3}\right]_t^6$ [by integration by parts with $u=r$]

$\displaystyle = \lim_{t\to-\infty}\left(18e^2-9e^2-3te^{t/3}+9e^{t/3}\right) = 9e^2-0+0$ [by l'Hospital's Rule]

$= 9e^2.$ Convergent

21. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t$ [by substitution with $u=\ln x$, $du = dx/x$] $\displaystyle = \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty.$ Divergent

22. $I = \int_{-\infty}^{\infty} x^3e^{-x^4}\,dx = I_1 + I_2 = \int_{-\infty}^{0} x^3e^{-x^4}\,dx + \int_0^\infty x^3e^{-x^4}\,dx$. Now

$\displaystyle I_2 = \lim_{t\to\infty}\int_0^t x^3e^{-x^4}\,dx = \lim_{t\to\infty}\int_0^{t^4} e^{-u}\left(\tfrac{1}{4}\,du\right)$ $\left[\begin{array}{c} u=x^4, \\ du = 4x^3\,dx \end{array}\right]$

$\displaystyle = \tfrac{1}{4}\lim_{t\to\infty}\left[-e^{-u}\right]_0^{t^4} = \tfrac{1}{4}\lim_{t\to\infty}\left(-e^{-t^4}+1\right) = \tfrac{1}{4}(0+1) = \tfrac{1}{4}.$

Since $f(x) = x^3e^{-x^4}$ is an odd function, $I_1 = -\frac{1}{4}$, and hence, $I = 0$. Convergent

23. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^{0} \frac{x^2}{9+x^6}\,dx + \int_{0}^{\infty} \frac{x^2}{9+x^6}\,dx = 2\int_{0}^{\infty} \frac{x^2}{9+x^6}\,dx$ [since the integrand is even].

Now $\displaystyle\int \frac{x^2\,dx}{9+x^6} \quad \begin{bmatrix} u = x^3 \\ du = 3x^2\,dx \end{bmatrix} \quad = \int \frac{\frac{1}{3}\,du}{9+u^2} \quad \begin{bmatrix} u = 3v \\ du = 3\,dv \end{bmatrix} \quad = \int \frac{\frac{1}{3}(3\,dv)}{9+9v^2} = \frac{1}{9}\int \frac{dv}{1+v^2}$

$$= \frac{1}{9}\tan^{-1} v + C = \frac{1}{9}\tan^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right) + C,$$

so $\displaystyle 2\int_0^{\infty} \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\int_0^t \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\left[\frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)\right]_0^t = 2\lim_{t\to\infty}\frac{1}{9}\tan^{-1}\left(\frac{t^3}{3}\right) = \frac{2}{9}\cdot\frac{\pi}{2} = \frac{\pi}{9}.$

Convergent

24. $\displaystyle\int_0^{\infty} \frac{e^x}{e^{2x}+3}\,dx = \lim_{t\to\infty}\int_0^t \frac{e^x}{(e^x)^2+(\sqrt{3})^2}\,dx = \lim_{t\to\infty}\left[\frac{1}{\sqrt{3}}\arctan\frac{e^x}{\sqrt{3}}\right]_0^t = \frac{1}{\sqrt{3}}\lim_{t\to\infty}\left(\arctan\frac{e^t}{\sqrt{3}} - \arctan\frac{1}{\sqrt{3}}\right)$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3}\right) = \frac{\pi\sqrt{3}}{9}.$$ Convergent

25. $\displaystyle\int_e^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_1^{\ln t} u^{-3}\,du \quad \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix} \quad = \lim_{t\to\infty}\left[-\frac{1}{2u^2}\right]_1^{\ln t}$

$$= \lim_{t\to\infty}\left[-\frac{1}{2(\ln t)^2}+\frac{1}{2}\right] = 0 + \frac{1}{2} = \frac{1}{2}.$$ Convergent

26. $\displaystyle\int_0^{\infty} \frac{x\arctan x}{(1+x^2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x\arctan x}{(1+x^2)^2}\,dx$. Let $u = \arctan x$, $\displaystyle dv = \frac{x\,dx}{(1+x^2)^2}$. Then $\displaystyle du = \frac{dx}{1+x^2}$,

$\displaystyle v = \frac{1}{2}\int \frac{2x\,dx}{(1+x^2)^2} = \frac{-1/2}{1+x^2}$, and

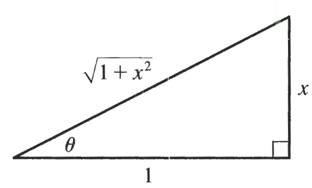
$$\int \frac{x\arctan x}{(1+x^2)^2}\,dx = -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{2}\int \frac{dx}{(1+x^2)^2} \quad \begin{bmatrix} x = \tan\theta, \\ dx = \sec^2\theta\,d\theta \end{bmatrix}$$

$$= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{2}\int \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2}$$

$$= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{2}\int \cos^2\theta\,d\theta$$

$$= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{\theta}{4} + \frac{\sin\theta\,\cos\theta}{4} + C$$

$$= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{4}\arctan x + \frac{1}{4}\frac{x}{1+x^2} + C$$



It follows that

$$\int_0^{\infty} \frac{x\arctan x}{(1+x^2)^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{4}\arctan x + \frac{1}{4}\frac{x}{1+x^2}\right]_0^t$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}\frac{\arctan t}{1+t^2} + \frac{1}{4}\arctan t + \frac{1}{4}\frac{t}{1+t^2}\right) = 0 + \frac{1}{4}\cdot\frac{\pi}{2} + 0 = \frac{\pi}{8}.$$ Convergent

27. $\displaystyle\int_0^1 \frac{3}{x^5}\,dx = \lim_{t\to 0^+}\int_t^1 3x^{-5}\,dx = \lim_{t\to 0^+}\left[-\frac{3}{4x^4}\right]_t^1 = -\frac{3}{4}\lim_{t\to 0^+}\left(1-\frac{1}{t^4}\right) = \infty.$ Divergent

28. $\displaystyle\int_2^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-}\int_2^t (3-x)^{-1/2}\,dx = \lim_{t\to 3^-}\left[-2(3-x)^{1/2}\right]_2^t = -2\lim_{t\to 3^-}\left(\sqrt{3-t}-\sqrt{1}\right) = -2(0-1) = 2.$

Convergent

29. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}} = \lim_{t\to -2^+}\int_t^{14} (x+2)^{-1/4}\,dx = \lim_{t\to -2^+}\left[\frac{4}{3}(x+2)^{3/4}\right]_t^{14} = \frac{4}{3}\lim_{t\to -2^+}\left[16^{3/4}-(t+2)^{3/4}\right]$

$= \frac{4}{3}(8-0) = \frac{32}{3}.$ Convergent

30. $\displaystyle\int_6^8 \frac{4}{(x-6)^3}\,dx = \lim_{t\to 6^+}\int_t^8 4(x-6)^{-3}\,dx = \lim_{t\to 6^+}\left[-2(x-6)^{-2}\right]_t^8 = -2\lim_{t\to 6^+}\left[\frac{1}{2^2}-\frac{1}{(t-6)^2}\right] = \infty.$ Divergent

31. $\displaystyle\int_{-2}^3 \frac{dx}{x^4} = \int_{-2}^0 \frac{dx}{x^4} + \int_0^3 \frac{dx}{x^4}$, but $\displaystyle\int_{-2}^0 \frac{dx}{x^4} = \lim_{t\to 0^-}\left[-\frac{x^{-3}}{3}\right]_{-2}^t = \lim_{t\to 0^-}\left[-\frac{1}{3t^3}-\frac{1}{24}\right] = \infty.$ Divergent

32. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\left[\sin^{-1}x\right]_0^t = \lim_{t\to 1^-}\sin^{-1}t = \frac{\pi}{2}.$ Convergent

33. There is an infinite discontinuity at $x=1$. $\int_0^{33}(x-1)^{-1/5}\,dx = \int_0^1 (x-1)^{-1/5}\,dx + \int_1^{33}(x-1)^{-1/5}\,dx$. Here

$\int_0^1 (x-1)^{-1/5}\,dx = \lim_{t\to 1^-}\int_0^t (x-1)^{-1/5}\,dx = \lim_{t\to 1^-}\left[\frac{5}{4}(x-1)^{4/5}\right]_0^t = \lim_{t\to 1^-}\left[\frac{5}{4}(t-1)^{4/5}-\frac{5}{4}\right] = -\frac{5}{4}$ and

$\int_1^{33}(x-1)^{-1/5}\,dx = \lim_{t\to 1^+}\int_t^{33}(x-1)^{-1/5}\,dx = \lim_{t\to 1^+}\left[\frac{5}{4}(x-1)^{4/5}\right]_t^{33} = \lim_{t\to 1^+}\left[\frac{5}{4}\cdot 16 - \frac{5}{4}(t-1)^{4/5}\right] = 20.$

Thus, $\int_0^{33}(x-1)^{-1/5}\,dx = -\frac{5}{4}+20 = \frac{75}{4}.$ Convergent

34. $f(y) = 1/(4y-1)$ has an infinite discontinuity at $y=\frac{1}{4}$.

$\displaystyle\int_{1/4}^1 \frac{1}{4y-1}\,dy = \lim_{t\to (1/4)^+}\int_t^1 \frac{1}{4y-1}\,dy = \lim_{t\to (1/4)^+}\left[\tfrac{1}{4}\ln|4y-1|\right]_t^1 = \lim_{t\to (1/4)^+}\left[\tfrac{1}{4}\ln 3 - \tfrac{1}{4}\ln(4t-1)\right] = \infty,$

so $\displaystyle\int_{1/4}^1 \frac{1}{4y-1}\,dy$ diverges, and hence, $\displaystyle\int_0^1 \frac{1}{4y-1}\,dy$ diverges. Divergent

35. $\displaystyle I = \int_0^3 \frac{dx}{x^2-6x+5} = \int_0^3 \frac{dx}{(x-1)(x-5)} = I_1 + I_2 = \int_0^1 \frac{dx}{(x-1)(x-5)} + \int_1^3 \frac{dx}{(x-1)(x-5)}.$

Now $\displaystyle\frac{1}{(x-1)(x-5)} = \frac{A}{x-1}+\frac{B}{x-5} \;\Rightarrow\; 1 = A(x-5)+B(x-1).$

Set $x=5$ to get $1=4B$, so $B=\frac{1}{4}$. Set $x=1$ to get $1=-4A$, so $A=-\frac{1}{4}$. Thus

$$I_1 = \lim_{t\to 1^-}\int_0^t \left(\frac{-\frac{1}{4}}{x-1}+\frac{\frac{1}{4}}{x-5}\right)dx = \lim_{t\to 1^-}\left[-\frac{1}{4}\ln|x-1|+\frac{1}{4}\ln|x-5|\right]_0^t$$

$$= \lim_{t\to 1^-}\left[\left(-\tfrac{1}{4}\ln|t-1|+\tfrac{1}{4}\ln|t-5|\right)-\left(-\tfrac{1}{4}\ln|-1|+\tfrac{1}{4}\ln|-5|\right)\right]$$

$$= \infty, \text{ since } \lim_{t\to 1^-}\left(-\tfrac{1}{4}\ln|t-1|\right) = \infty.$$

Since I_1 is divergent, I is divergent.

36. $\int_{\pi/2}^{\pi}\csc x\,dx = \lim_{t\to\pi^-}\int_{\pi/2}^t \csc x\,dx = \lim_{t\to\pi^-}\left[\ln|\csc x-\cot x|\right]_{\pi/2}^t = \lim_{t\to\pi^-}\left[\ln(\csc t-\cot t)-\ln(1-0)\right]$

$\displaystyle= \lim_{t\to\pi^-}\ln\left(\frac{1-\cos t}{\sin t}\right) = \infty.$ Divergent

37. $\displaystyle\int_{-1}^{0} \frac{e^{1/x}}{x^3}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x} e^{1/x} \cdot \frac{1}{x^2}\,dx = \lim_{t\to 0^-} \int_{-1}^{1/t} ue^u\,(-du) \qquad \left[\begin{array}{l} u = 1/x, \\ du = -dx/x^2 \end{array}\right]$

$\displaystyle = \lim_{t\to 0^-} \left[(u-1)e^u\right]_{1/t}^{-1} \qquad \left[\begin{array}{l}\text{use parts}\\ \text{or Formula 96}\end{array}\right] \quad = \lim_{t\to 0^-}\left[-2e^{-1} - \left(\frac{1}{t} - 1\right)e^{1/t}\right]$

$\displaystyle = -\frac{2}{e} - \lim_{s\to -\infty}(s-1)e^s \quad [s = 1/t] \quad = -\frac{2}{e} - \lim_{s\to-\infty}\frac{s-1}{e^{-s}} \overset{\text{H}}{=} -\frac{2}{e} - \lim_{s\to-\infty}\frac{1}{-e^{-s}}$

$\displaystyle = -\frac{2}{e} - 0 = -\frac{2}{e}.$ Convergent

38. $\displaystyle\int_{0}^{1} \frac{e^{1/x}}{x^3}\,dx = \lim_{t\to 0^+} \int_{t}^{1} \frac{1}{x} e^{1/x} \cdot \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_{1/t}^{1} ue^u\,(-du) \qquad \left[\begin{array}{l} u = 1/x, \\ du = -dx/x^2 \end{array}\right]$

$\displaystyle = \lim_{t\to 0^+} \left[(u-1)e^u\right]_{1}^{1/t} \qquad \left[\begin{array}{l}\text{use parts}\\ \text{or Formula 96}\end{array}\right] = \lim_{t\to 0^+}\left[\left(\frac{1}{t} - 1\right)e^{1/t} - 0\right]$

$\displaystyle = \lim_{s\to\infty}(s-1)e^s \quad [s = 1/t] \quad = \infty.$ Divergent

39. $\displaystyle I = \int_0^2 z^2 \ln z\,dz = \lim_{t\to 0^+}\int_t^2 z^2\ln z\,dz = \lim_{t\to 0^+}\left[\frac{z^3}{3^2}(3\ln z - 1)\right]_t^2 \qquad \left[\begin{array}{l}\text{integrate by parts}\\ \text{or use Formula 101}\end{array}\right]$

$\displaystyle = \lim_{t\to 0^+}\left[\tfrac{8}{9}(3\ln 2 - 1) - \tfrac{1}{9}t^3(3\ln t - 1)\right] = \tfrac{8}{3}\ln 2 - \tfrac{8}{9} - \tfrac{1}{9}\lim_{t\to 0^+}\left[t^3(3\ln t - 1)\right] = \tfrac{8}{3}\ln 2 - \tfrac{8}{9} - \tfrac{1}{9}L.$

Now $\displaystyle L = \lim_{t\to 0^+}\left[t^3(3\ln t - 1)\right] = \lim_{t\to 0^+}\frac{3\ln t - 1}{t^{-3}} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{3/t}{-3/t^4} = \lim_{t\to 0^+}\left(-t^3\right) = 0.$

Thus, $L = 0$ and $I = \tfrac{8}{3}\ln 2 - \tfrac{8}{9}$. Convergent

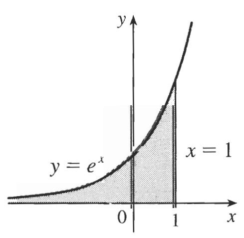
40. Integrate by parts with $u = \ln x$, $dv = dx/\sqrt{x} \quad\Rightarrow\quad du = dx/x$, $v = 2\sqrt{x}$.

$\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left(\left[2\sqrt{x}\ln x\right]_t^1 - 2\int_t^1 \frac{dx}{\sqrt{x}}\right) = \lim_{t\to 0^+}\left(-2\sqrt{t}\,\ln t - 4\left[\sqrt{x}\right]_t^1\right)$

$\displaystyle = \lim_{t\to 0^+}\left(-2\sqrt{t}\,\ln t - 4 + 4\sqrt{t}\right) = -4$

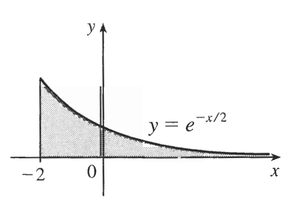
since $\displaystyle\lim_{t\to 0^+}\sqrt{t}\,\ln t = \lim_{t\to 0^+}\frac{\ln t}{t^{-1/2}} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{1/t}{-t^{-3/2}/2} = \lim_{t\to 0^+}\left(-2\sqrt{t}\right) = 0.$ Convergent

41.



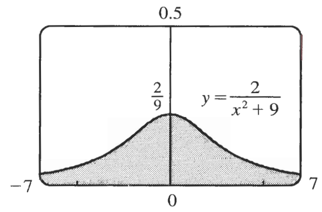
$\displaystyle \text{Area} = \int_{-\infty}^{1} e^x\,dx = \lim_{t\to-\infty}\left[e^x\right]_t^1 = e - \lim_{t\to-\infty} e^t = e$

42.



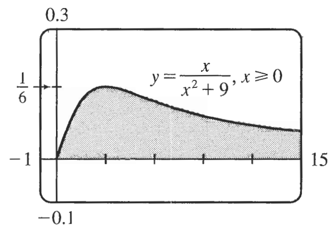
$\displaystyle \text{Area} = \int_{-2}^{\infty} e^{-x/2}\,dx = -2\lim_{t\to\infty}\left[e^{-x/2}\right]_{-2}^{t} = -2\lim_{t\to\infty}e^{-t/2} + 2e = 2e$

43.



$$\text{Area} = \int_{-\infty}^{\infty} \frac{2}{x^2+9}\,dx = 2\cdot 2\int_0^{\infty} \frac{1}{x^2+9}\,dx = 4\lim_{t\to\infty}\int_0^t \frac{1}{x^2+9}\,dx$$
$$= 4\lim_{t\to\infty}\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^t = \frac{4}{3}\lim_{t\to\infty}\left[\tan^{-1}\frac{t}{3} - 0\right] = \frac{4}{3}\cdot\frac{\pi}{2} = \frac{2\pi}{3}$$

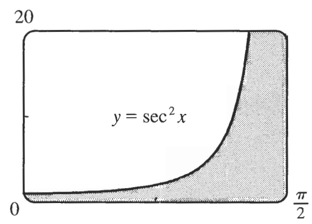
44.



$$\text{Area} = \int_0^{\infty} \frac{x}{x^2+9}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{x^2+9}\,dx = \lim_{t\to\infty}\left[\tfrac{1}{2}\ln\left(x^2+9\right)\right]_0^t$$
$$= \tfrac{1}{2}\lim_{t\to\infty}\left[\ln\left(t^2+9\right) - \ln 9\right] = \infty$$

Infinite area

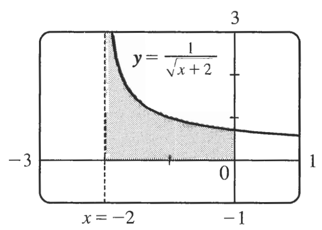
45.



$$\text{Area} = \int_0^{\pi/2} \sec^2 x\,dx = \lim_{t\to(\pi/2)^-}\int_0^t \sec^2 x\,dx = \lim_{t\to(\pi/2)^-}\left[\tan x\right]_0^t$$
$$= \lim_{t\to(\pi/2)^-}(\tan t - 0) = \infty$$

Infinite area

46.



$$\text{Area} = \int_{-2}^{0} \frac{1}{\sqrt{x+2}}\,dx = \lim_{t\to -2^+}\int_t^0 \frac{1}{\sqrt{x+2}}\,dx = \lim_{t\to -2^+}\left[2\sqrt{x+2}\right]_t^0$$
$$= \lim_{t\to -2^+}\left(2\sqrt{2} - 2\sqrt{t+2}\right) = 2\sqrt{2} - 0 = 2\sqrt{2}$$

47. (a)

t	$\int_1^t g(x)\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10,000	0.673407

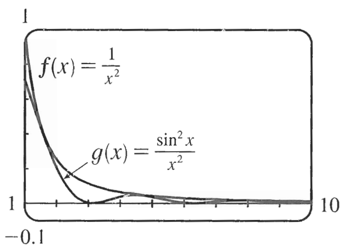
$g(x) = \dfrac{\sin^2 x}{x^2}$.

It appears that the integral is convergent.

(b) $-1 \le \sin x \le 1 \;\Rightarrow\; 0 \le \sin^2 x \le 1 \;\Rightarrow\; 0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$. Since $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ is convergent

[Equation 2 with $p = 2 > 1$], $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$ is convergent by the Comparison Theorem.

(c)



Since $\int_1^\infty f(x)\,dx$ is finite and the area under $g(x)$ is less than the area under $f(x)$ on any interval $[1, t]$, $\int_1^\infty g(x)\,dx$ must be finite; that is, the integral is convergent.

48. (a)

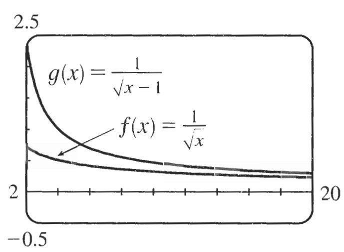
t	$\int_2^t g(x)dx$
5	3.830327
10	6.801200
100	23.328769
1000	69.023361
10,000	208.124560

$g(x) = \dfrac{1}{\sqrt{x} - 1}$.

It appears that the integral is divergent.

(b) For $x \geq 2$, $\sqrt{x} > \sqrt{x} - 1 \;\Rightarrow\; \dfrac{1}{\sqrt{x}} < \dfrac{1}{\sqrt{x} - 1}$. Since $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent [Equation 2 with $p = \frac{1}{2} \leq 1$], $\displaystyle\int_2^\infty \frac{1}{\sqrt{x} - 1}\,dx$ is divergent by the Comparison Theorem.

(c)



Since $\int_2^\infty f(x)\,dx$ is infinite and the area under $g(x)$ is greater than the area under $f(x)$ on any interval $[2, t]$, $\int_2^\infty g(x)\,dx$ must be infinite; that is, the integral is divergent.

49. For $x > 0$, $\dfrac{x}{x^3 + 1} < \dfrac{x}{x^3} = \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Equation 2 with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{x}{x^3 + 1}\,dx$ is convergent by the Comparison Theorem. $\displaystyle\int_0^1 \frac{x}{x^3 + 1}\,dx$ is a constant, so $\displaystyle\int_0^\infty \frac{x}{x^3 + 1}\,dx = \int_0^1 \frac{x}{x^3 + 1}\,dx + \int_1^\infty \frac{x}{x^3 + 1}\,dx$ is also convergent.

50. For $x \geq 1$, $\dfrac{2 + e^{-x}}{x} > \dfrac{2}{x}$ [since $e^{-x} > 0$] $> \dfrac{1}{x}$. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is divergent by Equation 2 with $p = 1 \leq 1$, so $\displaystyle\int_1^\infty \frac{2 + e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem.

51. For $x > 1$, $f(x) = \dfrac{x + 1}{\sqrt{x^4 - x}} > \dfrac{x + 1}{\sqrt{x^4}} > \dfrac{x}{x^2} = \dfrac{1}{x}$, so $\displaystyle\int_2^\infty f(x)\,dx$ diverges by comparison with $\displaystyle\int_2^\infty \frac{1}{x}\,dx$, which diverges by Equation 2 with $p = 1 \leq 1$. Thus, $\int_1^\infty f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^\infty f(x)\,dx$ also diverges.

52. For $x \geq 0$, $\arctan x < \frac{\pi}{2} < 2$, so $\dfrac{\arctan x}{2+e^x} < \dfrac{2}{2+e^x} < \dfrac{2}{e^x} = 2e^{-x}$. Now

$$I = \int_0^\infty 2e^{-x}\,dx = \lim_{t\to\infty}\int_0^t 2e^{-x}\,dx = \lim_{t\to\infty}\left[-2e^{-x}\right]_0^t = \lim_{t\to\infty}\left(-\frac{2}{e^t}+2\right) = 2,$$ so I is convergent, and by comparison,

$\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$ is convergent.

53. For $0 < x \leq 1$, $\dfrac{\sec^2 x}{x\sqrt{x}} > \dfrac{1}{x^{3/2}}$. Now

$$I = \int_0^1 x^{-3/2}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-3/2}\,dx = \lim_{t\to 0^+}\left[-2x^{-1/2}\right]_t^1 = \lim_{t\to 0^+}\left(-2+\frac{2}{\sqrt{t}}\right) = \infty,$$ so I is divergent, and by

comparison, $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}$ is divergent.

54. For $0 < x \leq 1$, $\dfrac{\sin^2 x}{\sqrt{x}} \leq \dfrac{1}{\sqrt{x}}$. Now

$$I = \int_0^\pi \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^\pi x^{-1/2}\,dx = \lim_{t\to 0^+}\left[2x^{1/2}\right]_t^\pi = \lim_{t\to 0^+}\left(2\pi - 2\sqrt{t}\right) = 2\pi - 0 = 2\pi,$$ so I is convergent, and by

comparison, $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$ is convergent.

55. $$\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}\,(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{x}\,(1+x)} + \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}\,(1+x)}.$$ Now

$$\int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2u\,du}{u(1+u^2)} \quad \begin{bmatrix} u=\sqrt{x},\ x=u^2, \\ dx = 2u\,du \end{bmatrix} \quad = 2\int \frac{du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C,$$ so

$$\begin{aligned}\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} &= \lim_{t\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^t \\ &= \lim_{t\to 0^+}\left[2\left(\tfrac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\right] + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2\left(\tfrac{\pi}{4}\right)\right] = \tfrac{\pi}{2} - 0 + 2\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2} = \pi.\end{aligned}$$

56. $$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+}\int_t^3 \frac{dx}{x\sqrt{x^2-4}} + \lim_{t\to\infty}\int_3^t \frac{dx}{x\sqrt{x^2-4}}.$$ Now

$$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{2\sec\theta\,\tan\theta\,d\theta}{2\sec\theta\,2\tan\theta} \quad \begin{bmatrix} x = 2\sec\theta, \text{ where} \\ 0\leq\theta<\pi/2 \text{ or } \pi\leq\theta<3\pi/2 \end{bmatrix} \quad = \tfrac{1}{2}\theta + C = \tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right) + C,$$ so

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+}\left[\tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right)\right]_t^3 + \lim_{t\to\infty}\left[\tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right)\right]_3^t = \tfrac{1}{2}\sec^{-1}\left(\tfrac{3}{2}\right) - 0 + \tfrac{1}{2}\left(\tfrac{\pi}{2}\right) - \tfrac{1}{2}\sec^{-1}\left(\tfrac{3}{2}\right) = \tfrac{\pi}{4}.$$

57. If $p = 1$, then $\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x} = \lim_{t\to 0^+} [\ln x]_t^1 = \infty$. Divergent.

If $p \neq 1$, then $\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^p}$ [note that the integral is not improper if $p < 0$]

$$= \lim_{t\to 0^+} \left[\frac{x^{-p+1}}{-p+1}\right]_t^1 = \lim_{t\to 0^+} \frac{1}{1-p}\left[1 - \frac{1}{t^{p-1}}\right]$$

If $p > 1$, then $p - 1 > 0$, so $\frac{1}{t^{p-1}} \to \infty$ as $t \to 0^+$, and the integral diverges.

If $p < 1$, then $p - 1 < 0$, so $\frac{1}{t^{p-1}} \to 0$ as $t \to 0^+$ and $\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}\left[\lim_{t\to 0^+}\left(1 - t^{1-p}\right)\right] = \frac{1}{1-p}$.

Thus, the integral converges if and only if $p < 1$, and in that case its value is $\frac{1}{1-p}$.

58. Let $u = \ln x$. Then $du = dx/x \;\Rightarrow\; \int_e^\infty \frac{dx}{x(\ln x)^p} = \int_1^\infty \frac{du}{u^p}$. By Example 4, this converges to $\frac{1}{p-1}$ if $p > 1$ and diverges otherwise.

59. First suppose $p = -1$. Then

$\int_0^1 x^p \ln x\,dx = \int_0^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+} \left[\frac{1}{2}(\ln x)^2\right]_t^1 = -\frac{1}{2}\lim_{t\to 0^+}(\ln t)^2 = -\infty$, so the integral diverges. Now suppose $p \neq -1$. Then integration by parts gives

$\int x^p \ln x\,dx = \frac{x^{p+1}}{p+1}\ln x - \int \frac{x^p}{p+1}\,dx = \frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2} + C$. If $p < -1$, then $p + 1 < 0$, so

$$\int_0^1 x^p \ln x\,dx = \lim_{t\to 0^+}\left[\frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2}\right]_t^1 = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\left[t^{p+1}\left(\ln t - \frac{1}{p+1}\right)\right] = \infty.$$

If $p > -1$, then $p + 1 > 0$ and

$$\int_0^1 x^p \ln x\,dx = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{\ln t - 1/(p+1)}{t^{-(p+1)}} \overset{\text{H}}{=} \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{1/t}{-(p+1)t^{-(p+2)}}$$

$$= \frac{-1}{(p+1)^2} + \frac{1}{(p+1)^2}\lim_{t\to 0^+} t^{p+1} = \frac{-1}{(p+1)^2}$$

Thus, the integral converges to $-\frac{1}{(p+1)^2}$ if $p > -1$ and diverges otherwise.

60. (a) $n = 0$: $\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}\left[-e^{-t} + 1\right] = 0 + 1 = 1$

$n = 1$: $\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x e^{-x}\,dx$. To evaluate $\int x e^{-x}\,dx$, we'll use integration by parts with $u = x$, $dv = e^{-x}\,dx \;\Rightarrow\; du = dx$, $v = -e^{-x}$.

So $\int x e^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} + C = (-x-1)e^{-x} + C$ and

$$\lim_{t\to\infty}\int_0^t x e^{-x}\,dx = \lim_{t\to\infty}\left[(-x-1)e^{-x}\right]_0^t = \lim_{t\to\infty}\left[(-t-1)e^{-t} + 1\right] = \lim_{t\to\infty}\left[-te^{-t} - e^{-t} + 1\right]$$

$$= 0 - 0 + 1 \quad [\text{use l'Hospital's Rule}] \quad = 1$$

$n = 2$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^2 e^{-x}\,dx$. To evaluate $\displaystyle\int x^2 e^{-x}\,dx$, we could use integration by parts again or Formula 97. Thus,

$$\lim_{t\to\infty}\int_0^t x^2 e^{-x}\,dx = \lim_{t\to\infty}\left[-x^2 e^{-x}\right]_0^t + 2\lim_{t\to\infty}\int_0^t x e^{-x}\,dx$$

$$= 0 + 0 + 2(1) \quad [\text{use l'Hospital's Rule and the result for } n = 1] \quad = 2$$

$n = 3$: $$\int_0^\infty x^n e^{-x} dx = \lim_{t\to\infty}\int_0^t x^3 e^{-x}\,dx \overset{97}{=} \lim_{t\to\infty}\left[-x^3 e^{-x}\right]_0^t + 3\lim_{t\to\infty}\int_0^t x^2 e^{-x}\,dx$$

$$= 0 + 0 + 3(2) \quad [\text{use l'Hospital's Rule and the result for } n = 2] \quad = 6$$

(b) For $n = 1$, 2, and 3, we have $\int_0^\infty x^n e^{-x}\,dx = 1$, 2, and 6. The values for the integral are equal to the factorials for n, so we guess $\int_0^\infty x^n e^{-x}\,dx = n!$.

(c) Suppose that $\int_0^\infty x^k e^{-x}\,dx = k!$ for some positive integer k. Then $\int_0^\infty x^{k+1} e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}\,dx$.

To evaluate $\int x^{k+1}e^{-x}\,dx$, we use parts with $u = x^{k+1}$, $dv = e^{-x}\,dx \;\Rightarrow\; du = (k+1)x^k\,dx$, $v = -e^{-x}$.

So $\int x^{k+1}e^{-x}dx = -x^{k+1}e^{-x} - \int -(k+1)x^k e^{-x}\,dx = -x^{k+1}e^{-x} + (k+1)\int x^k e^{-x}\,dx$ and

$$\lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}\,dx = \lim_{t\to\infty}\left[-x^{k+1}e^{-x}\right]_0^t + (k+1)\lim_{t\to\infty}\int_0^t x^k e^{-x}\,dx$$

$$= \lim_{t\to\infty}\left[-t^{k+1}e^{-t} + 0\right] + (k+1)k! = 0 + 0 + (k+1)! = (k+1)!,$$

so the formula holds for $k+1$. By induction, the formula holds for all positive integers. (Since $0! = 1$, the formula holds for $n = 0$, too.)

61. (a) $I = \int_{-\infty}^\infty x\,dx = \int_{-\infty}^0 x\,dx + \int_0^\infty x\,dx$, and $\int_0^\infty x\,dx = \lim_{t\to\infty}\int_0^t x\,dx = \lim_{t\to\infty}\left[\frac{1}{2}x^2\right]_0^t = \lim_{t\to\infty}\left[\frac{1}{2}t^2 - 0\right] = \infty$, so I is divergent.

(b) $\int_{-t}^t x\,dx = \left[\frac{1}{2}x^2\right]_{-t}^t = \frac{1}{2}t^2 - \frac{1}{2}t^2 = 0$, so $\lim_{t\to\infty}\int_{-t}^t x\,dx = 0$. Therefore, $\int_{-\infty}^\infty x\,dx \neq \lim_{t\to\infty}\int_{-t}^t x\,dx$.

62. Let $k = \dfrac{M}{2RT}$ so that $\overline{v} = \dfrac{4}{\sqrt{\pi}}k^{3/2}\displaystyle\int_0^\infty v^3 e^{-kv^2}\,dv$. Let I denote the integral and use parts to integrate I. Let $\alpha = v^2$, $d\beta = ve^{-kv^2}\,dv \;\Rightarrow\; d\alpha = 2v\,dv$, $\beta = -\dfrac{1}{2k}e^{-kv^2}$:

$$I = \lim_{t\to\infty}\left[-\frac{1}{2k}v^2 e^{-kv^2}\right]_0^t + \frac{1}{k}\int_0^\infty ve^{-kv^2}\,dv_0^t = -\frac{1}{2k}\lim_{t\to\infty}\left(t^2 e^{-kt^2}\right) + \frac{1}{k}\lim_{t\to\infty}\left[-\frac{1}{2k}e^{-kv^2}\right]$$

$$\overset{\text{H}}{=} -\frac{1}{2k}\cdot 0 - \frac{1}{2k^2}(0-1) = \frac{1}{2k^2}$$

Thus, $\overline{v} = \dfrac{4}{\sqrt{\pi}}k^{3/2}\cdot\dfrac{1}{2k^2} = \dfrac{2}{(k\pi)^{1/2}} = \dfrac{2}{[\pi M/(2RT)]^{1/2}} = \dfrac{2\sqrt{2}\sqrt{RT}}{\sqrt{\pi M}} = \sqrt{\dfrac{8RT}{\pi M}}$.

63. Volume $= \displaystyle\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi\lim_{t\to\infty}\int_1^t \frac{dx}{x^2} = \pi\lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \pi\lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = \pi < \infty.$

64. Work $= \int_R^\infty \frac{GMm}{r^2}\,dr = \lim_{t\to\infty}\int_R^t \frac{GMm}{r^2}\,dr = \lim_{t\to\infty} GMm\left[\frac{-1}{r}\right]_R^t = GMm \lim_{t\to\infty}\left(\frac{-1}{t}+\frac{1}{R}\right) = \frac{GMm}{R}$, where

M = mass of the earth $= 5.98\times 10^{24}$ kg, m = mass of satellite $= 10^3$ kg, R = radius of the earth $= 6.37\times 10^6$ m, and

G = gravitational constant $= 6.67\times 10^{-11}$ N·m^2/kg.

Therefore, Work $= \dfrac{6.67\times 10^{-11}\cdot 5.98\times 10^{24}\cdot 10^3}{6.37\times 10^6} \approx 6.26\times 10^{10}$ J.

65. Work $= \int_R^\infty F\,dr = \lim_{t\to\infty}\int_R^t \frac{GmM}{r^2}\,dr = \lim_{t\to\infty} GmM\left(\frac{1}{R}-\frac{1}{t}\right) = \frac{GmM}{R}$. The initial kinetic energy provides the work,

so $\frac{1}{2}mv_0^2 = \frac{GmM}{R} \;\Rightarrow\; v_0 = \sqrt{\frac{2GM}{R}}$.

66. $y(s) = \int_s^R \frac{2r}{\sqrt{r^2-s^2}}\,x(r)\,dr$ and $x(r) = \frac{1}{2}(R-r)^2 \;\Rightarrow$

$$\begin{aligned} y(s) &= \lim_{t\to s^+}\int_t^R \frac{r(R-r)^2}{\sqrt{r^2-s^2}}\,dr = \lim_{t\to s^+}\int_t^R \frac{r^3-2Rr^2+R^2r}{\sqrt{r^2-s^2}}\,dr \\ &= \lim_{t\to s^+}\left[\int_t^R \frac{r^3\,dr}{\sqrt{r^2-s^2}} - 2R\int_t^R \frac{r^2\,dr}{\sqrt{r^2-s^2}} + R^2\int_t^R \frac{r\,dr}{\sqrt{r^2-s^2}}\right] = \lim_{t\to s^+}\left(I_1 - 2RI_2 + R^2 I_3\right) = L \end{aligned}$$

For I_1: Let $u = \sqrt{r^2-s^2} \;\Rightarrow\; u^2 = r^2-s^2$, $r^2 = u^2+s^2$, $2r\,dr = 2u\,du$, so, omitting limits and constant of integration,

$$\begin{aligned} I_1 &= \int \frac{(u^2+s^2)u}{u}\,du = \int (u^2+s^2)\,du = \tfrac{1}{3}u^3 + s^2u = \tfrac{1}{3}u(u^2+3s^2) \\ &= \tfrac{1}{3}\sqrt{r^2-s^2}\,(r^2-s^2+3s^2) = \tfrac{1}{3}\sqrt{r^2-s^2}\,(r^2+2s^2) \end{aligned}$$

For I_2: Using Formula 44, $I_2 = \frac{r}{2}\sqrt{r^2-s^2} + \frac{s^2}{2}\ln\left|r+\sqrt{r^2-s^2}\right|$.

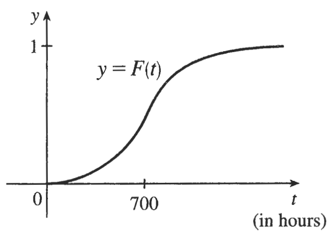
For I_3: Let $u = r^2-s^2 \;\Rightarrow\; du = 2r\,dr$. Then $I_3 = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\cdot 2\sqrt{u} = \sqrt{r^2-s^2}$.

Thus,

$$\begin{aligned} L &= \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{r^2-s^2}\,(r^2+2s^2) - 2R\left(\frac{r}{2}\sqrt{r^2-s^2} + \frac{s^2}{2}\ln\left|r+\sqrt{r^2-s^2}\right|\right) + R^2\sqrt{r^2-s^2}\right]_t^R \\ &= \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{R^2-s^2}(R^2+2s^2) - 2R\left(\frac{R}{2}\sqrt{R^2-s^2} + \frac{s^2}{2}\ln\left|R+\sqrt{R^2-s^2}\right|\right) + R^2\sqrt{R^2-s^2}\right] \\ &\qquad - \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{t^2-s^2}\,(t^2+2s^2) - 2R\left(\frac{t}{2}\sqrt{t^2-s^2} + \frac{s^2}{2}\ln\left|t+\sqrt{t^2-s^2}\right|\right) + R^2\sqrt{t^2-s^2}\right] \\ &= \left[\tfrac{1}{3}\sqrt{R^2-s^2}\,(R^2+2s^2) - Rs^2\ln\left|R+\sqrt{R^2-s^2}\right|\right] - \left[-Rs^2\ln|s|\right] \\ &= \tfrac{1}{3}\sqrt{R^2-s^2}\,(R^2+2s^2) - Rs^2\ln\left(\frac{R+\sqrt{R^2-s^2}}{s}\right) \end{aligned}$$

67. We would expect a small percentage of bulbs to burn out in the first few hundred hours, most of the bulbs to burn out after close to 700 hours, and a few overachievers to burn on and on.

(a)



(b) $r(t) = F'(t)$ is the rate at which the fraction $F(t)$ of burnt-out bulbs increases as t increases. This could be interpreted as a fractional burnout rate.

(c) $\int_0^\infty r(t)\,dt = \lim\limits_{x\to\infty} F(x) = 1$, since all of the bulbs will eventually burn out.

68. $I = \displaystyle\int_0^\infty te^{kt}\,dt = \lim_{s\to\infty}\left[\frac{1}{k^2}(kt-1)\,e^{kt}\right]_0^s$ [Formula 96, or parts] $= \displaystyle\lim_{s\to\infty}\left[\left(\frac{1}{k}se^{ks} - \frac{1}{k^2}e^{ks}\right) - \left(-\frac{1}{k^2}\right)\right]$.

Since $k < 0$ the first two terms approach 0 (you can verify that the first term does so with l'Hospital's Rule), so the limit is equal to $1/k^2$. Thus, $M = -kI = -k(1/k^2) = -1/k = -1/(-0.000121) \approx 8264.5$ years.

69. $I = \displaystyle\int_a^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\int_a^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\left[\tan^{-1}x\right]_a^t = \lim_{t\to\infty}\left(\tan^{-1}t - \tan^{-1}a\right) = \tfrac{\pi}{2} - \tan^{-1}a.$

$I < 0.001 \;\Rightarrow\; \tfrac{\pi}{2} - \tan^{-1}a < 0.001 \;\Rightarrow\; \tan^{-1}a > \tfrac{\pi}{2} - 0.001 \;\Rightarrow\; a > \tan\left(\tfrac{\pi}{2} - 0.001\right) \approx 1000.$

70. $f(x) = e^{-x^2}$ and $\Delta x = \frac{4-0}{8} = \frac{1}{2}$.

$\int_0^4 f(x)\,dx \approx S_8 = \frac{1}{2\cdot 3}[f(0) + 4f(0.5) + 2f(1) + \cdots + 2f(3) + 4f(3.5) + f(4)] \approx \frac{1}{6}(5.31717808) \approx 0.8862$

Now $x > 4 \;\Rightarrow\; -x\cdot x < -x\cdot 4 \;\Rightarrow\; e^{-x^2} < e^{-4x} \;\Rightarrow\; \int_4^\infty e^{-x^2}\,dx < \int_4^\infty e^{-4x}\,dx.$

$\int_4^\infty e^{-4x}\,dx = \lim\limits_{t\to\infty}\left[-\frac{1}{4}e^{-4x}\right]_4^t = -\frac{1}{4}\left(0 - e^{-16}\right) = 1/(4e^{16}) \approx 0.0000000281 < 0.0000001$, as desired.

71. (a) $F(s) = \displaystyle\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \lim_{n\to\infty}\left[-\frac{e^{-st}}{s}\right]_0^n = \lim_{n\to\infty}\left(\frac{e^{-sn}}{-s} + \frac{1}{s}\right)$. This converges to $\dfrac{1}{s}$ only if $s > 0$.

Therefore $F(s) = \dfrac{1}{s}$ with domain $\{s \mid s > 0\}$.

(b) $F(s) = \displaystyle\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^te^{-st}\,dt = \lim_{n\to\infty}\int_0^n e^{t(1-s)}\,dt = \lim_{n\to\infty}\left[\frac{1}{1-s}e^{t(1-s)}\right]_0^n$

$= \displaystyle\lim_{n\to\infty}\left(\frac{e^{(1-s)n}}{1-s} - \frac{1}{1-s}\right)$

This converges only if $1 - s < 0 \;\Rightarrow\; s > 1$, in which case $F(s) = \dfrac{1}{s-1}$ with domain $\{s \mid s > 1\}$.

(c) $F(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim\limits_{n\to\infty}\int_0^n te^{-st}\,dt$. Use integration by parts: let $u = t$, $dv = e^{-st}\,dt \;\Rightarrow\; du = dt$,

$v = -\dfrac{e^{-st}}{s}$. Then $F(s) = \displaystyle\lim_{n\to\infty}\left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^n = \lim_{n\to\infty}\left(\frac{-n}{se^{sn}} - \frac{1}{s^2e^{sn}} + 0 + \frac{1}{s^2}\right) = \frac{1}{s^2}$ only if $s > 0$.

Therefore, $F(s) = \dfrac{1}{s^2}$ and the domain of F is $\{s \mid s > 0\}$.

72. $0 \le f(t) \le Me^{at} \;\Rightarrow\; 0 \le f(t)e^{-st} \le Me^{at}e^{-st}$ for $t \ge 0$. Now use the Comparison Theorem:

$$\int_0^\infty Me^{at}e^{-st}\,dt = \lim_{n\to\infty} M\int_0^n e^{t(a-s)}\,dt = M\cdot\lim_{n\to\infty}\left[\frac{1}{a-s}e^{t(a-s)}\right]_0^n = M\cdot\lim_{n\to\infty}\frac{1}{a-s}\left[e^{n(a-s)}-1\right]$$

This is convergent only when $a-s<0 \;\Rightarrow\; s>a$. Therefore, by the Comparison Theorem, $F(s)=\int_0^\infty f(t)\,e^{-st}\,dt$ is also convergent for $s>a$.

73. $G(s)=\int_0^\infty f'(t)e^{-st}\,dt$. Integrate by parts with $u=e^{-st}$, $dv=f'(t)\,dt \;\Rightarrow\; du=-se^{-st}$, $v=f(t)$:

$$G(s)=\lim_{n\to\infty}\left[f(t)e^{-st}\right]_0^n + s\int_0^\infty f(t)e^{-st}\,dt = \lim_{n\to\infty} f(n)e^{-sn} - f(0) + sF(s)$$

But $0\le f(t)\le Me^{at} \;\Rightarrow\; 0\le f(t)e^{-st}\le Me^{at}e^{-st}$ and $\lim\limits_{t\to\infty} Me^{t(a-s)}=0$ for $s>a$. So by the Squeeze Theorem, $\lim\limits_{t\to\infty} f(t)e^{-st}=0$ for $s>a \;\Rightarrow\; G(s)=0-f(0)+sF(s)=sF(s)-f(0)$ for $s>a$.

74. Assume without loss of generality that $a<b$. Then

$$\begin{aligned}
\int_{-\infty}^a f(x)\,dx+\int_a^\infty f(x)\,dx &= \lim_{t\to-\infty}\int_t^a f(x)\,dx+\lim_{u\to\infty}\int_a^u f(x)\,dx\\
&= \lim_{t\to-\infty}\int_t^a f(x)\,dx+\lim_{u\to\infty}\left[\int_a^b f(x)\,dx+\int_b^u f(x)\,dx\right]\\
&= \lim_{t\to-\infty}\int_t^a f(x)\,dx+\int_a^b f(x)\,dx+\lim_{u\to\infty}\int_b^u f(x)\,dx\\
&= \lim_{t\to-\infty}\left[\int_t^a f(x)\,dx+\int_a^b f(x)\,dx\right]+\int_b^\infty f(x)\,dx\\
&= \lim_{t\to-\infty}\int_t^b f(x)\,dx+\int_b^\infty f(x)\,dx=\int_{-\infty}^b f(x)\,dx+\int_b^\infty f(x)\,dx
\end{aligned}$$

75. We use integration by parts: let $u=x$, $dv=xe^{-x^2}\,dx \;\Rightarrow\; du=dx$, $v=-\frac{1}{2}e^{-x^2}$. So

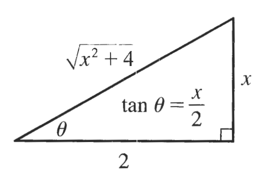
$$\int_0^\infty x^2e^{-x^2}\,dx=\lim_{t\to\infty}\left[-\frac{1}{2}xe^{-x^2}\right]_0^t+\frac{1}{2}\int_0^\infty e^{-x^2}\,dx=\lim_{t\to\infty}\left[-\frac{t}{2e^{t^2}}\right]+\frac{1}{2}\int_0^\infty e^{-x^2}\,dx=\frac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

(The limit is 0 by l'Hospital's Rule.)

76. $\int_0^\infty e^{-x^2}\,dx$ is the area under the curve $y=e^{-x^2}$ for $0\le x<\infty$ and $0<y\le 1$. Solving $y=e^{-x^2}$ for x, we get $y=e^{-x^2} \;\Rightarrow\; \ln y=-x^2 \;\Rightarrow\; -\ln y=x^2 \;\Rightarrow\; x=\pm\sqrt{-\ln y}$. Since x is positive, choose $x=\sqrt{-\ln y}$, and the area is represented by $\int_0^1\sqrt{-\ln y}\,dy$. Therefore, each integral represents the same area, so the integrals are equal.

77. For the first part of the integral, let $x=2\tan\theta \;\Rightarrow\; dx=2\sec^2\theta\,d\theta$.

$$\int\frac{1}{\sqrt{x^2+4}}\,dx=\int\frac{2\sec^2\theta}{2\sec\theta}\,d\theta=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|.$$



From the figure, $\tan\theta=\dfrac{x}{2}$, and $\sec\theta=\dfrac{\sqrt{x^2+4}}{2}$. So

$$I = \int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right) dx = \lim_{t\to\infty}\left[\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| - C\ln|x+2|\right]_0^t$$

$$= \lim_{t\to\infty}\left[\ln\frac{\sqrt{t^2+4}+t}{2} - C\ln(t+2) - (\ln 1 - C\ln 2)\right]$$

$$= \lim_{t\to\infty}\left[\ln\left(\frac{\sqrt{t^2+4}+t}{2\,(t+2)^C}\right) + \ln 2^C\right] = \ln\left(\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C}\right) + \ln 2^{C-1}$$

Now $L = \lim\limits_{t\to\infty}\dfrac{t+\sqrt{t^2+4}}{(t+2)^C} \overset{\text{H}}{=} \lim\limits_{t\to\infty}\dfrac{1+t/\sqrt{t^2+4}}{C\,(t+2)^{C-1}} = \dfrac{2}{C\lim\limits_{t\to\infty}(t+2)^{C-1}}$.

If $C < 1$, $L = \infty$ and I diverges.

If $C = 1$, $L = 2$ and I converges to $\ln 2 + \ln 2^0 = \ln 2$.

If $C > 1$, $L = 0$ and I diverges to $-\infty$.

78. $I = \displaystyle\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right) dx = \lim_{t\to\infty}\left[\tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{3}C\ln(3x+1)\right]_0^t = \lim_{t\to\infty}\left[\ln(t^2+1)^{1/2} - \ln(3t+1)^{C/3}\right]$

$$= \lim_{t\to\infty}\left(\ln\frac{(t^2+1)^{1/2}}{(3t+1)^{C/3}}\right) = \ln\left(\lim_{t\to\infty}\frac{\sqrt{t^2+1}}{(3t+1)^{C/3}}\right)$$

For $C \le 0$, the integral diverges. For $C > 0$, we have

$$L = \lim_{t\to\infty}\frac{\sqrt{t^2+1}}{(3t+1)^{C/3}} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{t/\sqrt{t^2+1}}{C(3t+1)^{(C/3)-1}} = \frac{1}{C}\lim_{t\to\infty}\frac{1}{(3t+1)^{(C/3)-1}}$$

For $C/3 < 1 \iff C < 3$, $L = \infty$ and I diverges.

For $C = 3$, $L = \frac{1}{3}$ and $I = \ln\frac{1}{3}$.

For $C > 3$, $L = 0$ and I diverges to $-\infty$.

79. No, $I = \int_0^\infty f(x)\,dx$ must be *divergent*. Since $\lim\limits_{x\to\infty} f(x) = 1$, there must exist an N such that if $x \ge N$, then $f(x) \ge \frac{1}{2}$.

Thus, $I = I_1 + I_2 = \int_0^N f(x)\,dx + \int_N^\infty f(x)\,dx$, where I_1 is an ordinary definite integral that has a finite value, and I_2 is improper and diverges by comparison with the divergent integral $\int_N^\infty \frac{1}{2}\,dx$.

80. As in Exercise 55, we let $I = \displaystyle\int_0^\infty \frac{x^a}{1+x^b}\,dx = I_1 + I_2$, where $I_1 = \displaystyle\int_0^1 \frac{x^a}{1+x^b}\,dx$ and $I_2 = \displaystyle\int_1^\infty \frac{x^a}{1+x^b}\,dx$. We will show that I_1 converges for $a > -1$ and I_2 converges for $b > a+1$, so that I converges when $a > -1$ and $b > a+1$.

I_1 is improper only when $a < 0$. When $0 \le x \le 1$, we have $\dfrac{1}{1+x^b} \le 1 \;\Rightarrow\; \dfrac{1}{x^{-a}(1+x^b)} \le \dfrac{1}{x^{-a}}$. The integral $\displaystyle\int_0^1 \frac{1}{x^{-a}}\,dx$ converges for $-a < 1$ [or $a > -1$] by Exercise 57, so by the Comparison Theorem, $\displaystyle\int_0^1 \frac{1}{x^{-a}(1+x^b)}\,dx$ converges for $-1 < a < 0$. I_1 is not improper when $a \ge 0$, so it has a finite real value in that case. Therefore, I_1 has a finite real value (converges) when $a > -1$.

I_2 is always improper. When $x \ge 1$, $\dfrac{x^a}{1+x^b} = \dfrac{1}{x^{-a}(1+x^b)} = \dfrac{1}{x^{-a}+x^{b-a}} < \dfrac{1}{x^{b-a}}$. By (2), $\displaystyle\int_1^\infty \frac{1}{x^{b-a}}\,dx$ converges for $b - a > 1$ (or $b > a+1$), so by the Comparison Theorem, $\displaystyle\int_1^\infty \frac{x^a}{1+x^b}\,dx$ converges for $b > a+1$.

Thus, I converges if $a > -1$ and $b > a+1$.