7 Review

CONCEPT CHECK

1. See Formula 7.1.1 or 7.1.2. We try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x)\,dx$ can be readily integrated to give v.

2. See the Strategy for Evaluating $\int \sin^m x \cos^n x\,dx$ on page 462.

3. If $\sqrt{a^2 - x^2}$ occurs, try $x = a\sin\theta$; if $\sqrt{a^2 + x^2}$ occurs, try $x = a\tan\theta$, and if $\sqrt{x^2 - a^2}$ occurs, try $x = a\sec\theta$. See the Table of Trigonometric Substitutions on page 467.

4. See Equation 2 and Expressions 7, 9, and 11 in Section 7.4.

5. See the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule, as well as their associated error bounds, all in Section 7.7. We would expect the best estimate to be given by Simpson's Rule.

6. See Definitions 1(a), (b), and (c) in Section 7.8.

7. See Definitions 3(b), (a), and (c) in Section 7.8.

8. See the Comparison Theorem after Example 8 in Section 7.8.

TRUE-FALSE QUIZ

1. False. Since the numerator has a higher degree than the denominator, $\dfrac{x(x^2+4)}{x^2-4} = x + \dfrac{8x}{x^2-4} = x + \dfrac{A}{x+2} + \dfrac{B}{x-2}$.

2. True. In fact, $A = -1$, $B = C = 1$.

3. False. It can be put in the form $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-4}$.

4. False. The form is $\dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$.

5. False. This is an improper integral, since the denominator vanishes at $x = 1$.

$$\int_0^4 \frac{x}{x^2-1}\,dx = \int_0^1 \frac{x}{x^2-1}\,dx + \int_1^4 \frac{x}{x^2-1}\,dx \text{ and}$$

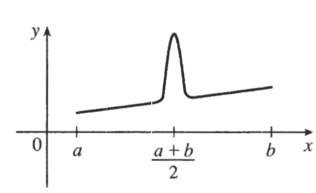
$$\int_0^1 \frac{x}{x^2-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{x}{x^2-1}\,dx = \lim_{t\to 1^-} \left[\tfrac{1}{2}\ln\left|x^2-1\right|\right]_0^t = \lim_{t\to 1^-} \tfrac{1}{2}\ln\left|t^2-1\right| = \infty$$

So the integral diverges.

6. True by Theorem 7.8.2 with $p = \sqrt{2} > 1$.

7. False. See Exercise 61 in Section 7.8.

8. False. For example, with $n = 1$ the Trapezoidal Rule is much more accurate than the Midpoint Rule for the function in the diagram.



9. (a) True. See the end of Section 7.5.

(b) False. Examples include the functions $f(x) = e^{x^2}$, $g(x) = \sin(x^2)$, and $h(x) = \dfrac{\sin x}{x}$.

10. True. If f is continuous on $[0, \infty)$, then $\int_0^1 f(x)\,dx$ is finite. Since $\int_1^\infty f(x)\,dx$ is finite, so is $\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx$.

11. False. If $f(x) = 1/x$, then f is continuous and decreasing on $[1, \infty)$ with $\lim\limits_{x\to\infty} f(x) = 0$, but $\int_1^\infty f(x)\,dx$ is divergent.

12. True.

$$\begin{aligned}\int_a^\infty [f(x) + g(x)]\,dx &= \lim_{t\to\infty} \int_a^t [f(x) + g(x)]\,dx = \lim_{t\to\infty} \left(\int_a^t f(x)\,dx + \int_a^t g(x)\,dx\right) \\ &= \lim_{t\to\infty} \int_a^t f(x)\,dx + \lim_{t\to\infty} \int_a^t g(x)\,dx \quad \left[\begin{array}{l}\text{since both limits} \\ \text{in the sum exist}\end{array}\right] \\ &= \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx\end{aligned}$$

Since the two integrals are finite, so is their sum.

13. False. Take $f(x) = 1$ for all x and $g(x) = -1$ for all x. Then $\int_a^\infty f(x)\,dx = \infty$ [divergent] and $\int_a^\infty g(x)\,dx = -\infty$ [divergent], but $\int_a^\infty [f(x) + g(x)]\,dx = 0$ [convergent].

14. False. $\int_0^\infty f(x)\,dx$ could converge or diverge. For example, if $g(x) = 1$, then $\int_0^\infty f(x)\,dx$ diverges if $f(x) = 1$ and converges if $f(x) = 0$.

EXERCISES

1. $$\begin{aligned}\int_0^5 \frac{x}{x+10}\,dx &= \int_0^5 \left(1 - \frac{10}{x+10}\right) dx = \Big[x - 10\ln(x+10)\Big]_0^5 = 5 - 10\ln 15 + 10\ln 10 \\ &= 5 + 10\ln\tfrac{10}{15} = 5 + 10\ln\tfrac{2}{3}\end{aligned}$$

2. $$\begin{aligned}\int_0^5 ye^{-0.6y}\,dy \quad \left[\begin{array}{ll}u = y, & dv = e^{-0.6y}\,dy, \\ du = dy & v = -\tfrac{5}{3}e^{-0.6y}\end{array}\right] &= \Big[-\tfrac{5}{3}ye^{-0.6y}\Big]_0^5 - \int_0^5 \left(-\tfrac{5}{3}e^{-0.6y}\right) dy = -\tfrac{25}{3}e^{-3} - \tfrac{25}{9}\Big[e^{-0.6y}\Big]_0^5 \\ &= -\tfrac{25}{3}e^{-3} - \tfrac{25}{9}(e^{-3} - 1) = -\tfrac{25}{3}e^{-3} - \tfrac{25}{9}e^{-3} + \tfrac{25}{9} = \tfrac{25}{9} - \tfrac{100}{9}e^{-3}\end{aligned}$$

3. $$\int_0^{\pi/2} \frac{\cos\theta}{1 + \sin\theta}\,d\theta = \Big[\ln(1 + \sin\theta)\Big]_0^{\pi/2} = \ln 2 - \ln 1 = \ln 2$$

4. $$\int_1^4 \frac{dt}{(2t+1)^3} \quad \left[\begin{array}{l}u = 2t+1, \\ du = 2\,dt\end{array}\right] = \int_3^9 \frac{\frac{1}{2}\,du}{u^3} = \frac{-1}{4}\left[\frac{1}{u^2}\right]_3^9 = -\frac{1}{4}\left(\frac{1}{81} - \frac{1}{9}\right) = -\frac{1}{4}\left(-\frac{8}{81}\right) = \frac{2}{81}$$

5. $$\begin{aligned}\int_0^{\pi/2} \sin^3\theta\,\cos^2\theta\,d\theta &= \int_0^{\pi/2} (1 - \cos^2\theta)\cos^2\theta\,\sin\theta\,d\theta = \int_1^0 (1 - u^2)u^2\,(-du) \quad \left[\begin{array}{l}u = \cos\theta, \\ du = -\sin\theta\,d\theta\end{array}\right] \\ &= \int_0^1 (u^2 - u^4)\,du = \Big[\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\Big]_0^1 = \left(\tfrac{1}{3} - \tfrac{1}{5}\right) - 0 = \tfrac{2}{15}\end{aligned}$$

6. $\dfrac{1}{y^2-4y-12}=\dfrac{1}{(y-6)(y+2)}=\dfrac{A}{y-6}+\dfrac{B}{y+2}\;\Rightarrow\; 1=A(y+2)+B(y-6)$. Letting $y=-2\;\Rightarrow\; B=-\frac{1}{8}$ and letting $y=6\;\Rightarrow\; A=\frac{1}{8}$. So $\displaystyle\int\frac{1}{y^2-4y-12}\,dy=\int\left(\frac{1/8}{y-6}+\frac{-1/8}{y+2}\right)dy=\tfrac{1}{8}\ln|y-6|-\tfrac{1}{8}\ln|y+2|+C$.

7. Let $u=\ln t$, $du=dt/t$. Then $\displaystyle\int\frac{\sin(\ln t)}{t}\,dt=\int\sin u\,du=-\cos u+C=-\cos(\ln t)+C$.

8. Let $u=\sqrt{e^x-1}$, so that $u^2=e^x-1$, $2u\,du=e^x\,dx$, and $e^x=u^2+1$. Then

$$\int\frac{1}{\sqrt{e^x-1}}\,dx=\int\frac{1}{u}\frac{2u\,du}{u^2+1}=2\int\frac{1}{u^2+1}\,du=2\tan^{-1}u+C=2\tan^{-1}\sqrt{e^x-1}+C.$$

9. $\displaystyle\int_1^4 x^{3/2}\ln x\,dx\quad\left[\begin{matrix}u=\ln x, & dv=x^{3/2}\,dx,\\ du=dx/x & v=\frac{2}{5}x^{5/2}\end{matrix}\right]=\frac{2}{5}\Big[x^{5/2}\ln x\Big]_1^4-\frac{2}{5}\int_1^4 x^{3/2}\,dx=\tfrac{2}{5}(32\ln 4-\ln 1)-\tfrac{2}{5}\Big[\tfrac{2}{5}x^{5/2}\Big]_1^4$

$$=\tfrac{2}{5}(64\ln 2)-\tfrac{4}{25}(32-1)=\tfrac{128}{5}\ln 2-\tfrac{124}{25}\quad\left[\text{or }\tfrac{64}{5}\ln 4-\tfrac{124}{25}\right]$$

10. Let $u=\arctan x$, $du=dx/(1+x^2)$. Then

$$\int_0^1\frac{\sqrt{\arctan x}}{1+x^2}\,dx=\int_0^{\pi/4}\sqrt{u}\,du=\frac{2}{3}\Big[u^{3/2}\Big]_0^{\pi/4}=\frac{2}{3}\left[\frac{\pi^{3/2}}{4^{3/2}}-0\right]=\frac{2}{3}\cdot\frac{1}{8}\pi^{3/2}=\frac{1}{12}\pi^{3/2}.$$

11. Let $x=\sec\theta$. Then

$$\int_1^2\frac{\sqrt{x^2-1}}{x}\,dx=\int_0^{\pi/3}\frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\,d\theta=\int_0^{\pi/3}\tan^2\theta\,d\theta=\int_0^{\pi/3}(\sec^2\theta-1)\,d\theta=\big[\tan\theta-\theta\big]_0^{\pi/3}=\sqrt{3}-\tfrac{\pi}{3}.$$

12. $\displaystyle\int_{-1}^1\frac{\sin x}{1+x^2}\,dx=0$ by Theorem 5.5.7(b), since $f(x)=\dfrac{\sin x}{1+x^2}$ is an odd function.

13. Let $t=\sqrt[3]{x}$. Then $t^3=x$ and $3t^2\,dt=dx$, so $\int e^{\sqrt[3]{x}}\,dx=\int e^t\cdot 3t^2\,dt=3I$. To evaluate I, let $u=t^2$, $dv=e^t\,dt\;\Rightarrow\; du=2t\,dt$, $v=e^t$, so $I=\int t^2e^t\,dt=t^2e^t-\int 2te^t\,dt$. Now let $U=t$, $dV=e^t\,dt\;\Rightarrow\; dU=dt$, $V=e^t$. Thus, $I=t^2e^t-2\big[te^t-\int e^t\,dt\big]=t^2e^t-2te^t+2e^t+C_1$, and hence

$3I=3e^t(t^2-2t+2)+C=3e^{\sqrt[3]{x}}(x^{2/3}-2x^{1/3}+2)+C$.

14. $\displaystyle\int\frac{x^2+2}{x+2}\,dx=\int\left(x-2+\frac{6}{x+2}\right)dx=\tfrac{1}{2}x^2-2x+6\ln|x+2|+C$

15. $\dfrac{x-1}{x^2+2x}=\dfrac{x-1}{x(x+2)}=\dfrac{A}{x}+\dfrac{B}{x+2}\;\Rightarrow\; x-1=A(x+2)+Bx$. Set $x=-2$ to get $-3=-2B$, so $B=\frac{3}{2}$. Set $x=0$ to get $-1=2A$, so $A=-\frac{1}{2}$. Thus, $\displaystyle\int\frac{x-1}{x^2+2x}\,dx=\int\left(\frac{-\frac{1}{2}}{x}+\frac{\frac{3}{2}}{x+2}\right)dx=-\frac{1}{2}\ln|x|+\frac{3}{2}\ln|x+2|+C$.

16. $\displaystyle\int\frac{\sec^6\theta}{\tan^2\theta}\,d\theta=\int\frac{(\tan^2\theta+1)^2\sec^2\theta}{\tan^2\theta}\,d\theta\quad\left[\begin{matrix}u=\tan\theta,\\ du=-\sec^2\theta\,d\theta\end{matrix}\right]=\int\frac{(u^2+1)^2}{u^2}\,du=\int\frac{u^4+2u^2+1}{u^2}\,du$

$$=\int\left(u^2+2+\frac{1}{u^2}\right)du=\frac{u^3}{3}+2u-\frac{1}{u}+C=\tfrac{1}{3}\tan^3\theta+2\tan\theta-\cot\theta+C$$

17. Integrate by parts with $u = x$, $dv = \sec x \, \tan x \, dx \;\Rightarrow\; du = dx$, $v = \sec x$:

$$\int x \sec x \, \tan x \, dx = x \sec x - \int \sec x \, dx \overset{14}{=} x \sec x - \ln|\sec x + \tan x| + C.$$

18. $\dfrac{x^2 + 8x - 3}{x^3 + 3x^2} = \dfrac{x^2 + 8x - 3}{x^2(x+3)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+3} \;\Rightarrow\; x^2 + 8x - 3 = Ax(x+3) + B(x+3) + Cx^2.$

Taking $x = 0$, we get $-3 = 3B$, so $B = -1$. Taking $x = -3$, we get $-18 = 9C$, so $C = -2$.

Taking $x = 1$, we get $6 = 4A + 4B + C = 4A - 4 - 2$, so $4A = 12$ and $A = 3$. Now

$$\int \frac{x^2 + 8x - 3}{x^3 + 3x^2} \, dx = \int \left(\frac{3}{x} - \frac{1}{x^2} - \frac{2}{x+3} \right) dx = 3 \ln|x| + \frac{1}{x} - 2 \ln|x+3| + C.$$

19. $$\int \frac{x+1}{9x^2 + 6x + 5} \, dx = \int \frac{x+1}{(9x^2 + 6x + 1) + 4} \, dx = \int \frac{x+1}{(3x+1)^2 + 4} \, dx \qquad \begin{bmatrix} u = 3x + 1, \\ du = 3\,dx \end{bmatrix}$$

$$= \int \frac{\left[\frac{1}{3}(u-1)\right] + 1}{u^2 + 4} \left(\frac{1}{3} \, du \right) = \frac{1}{3} \cdot \frac{1}{3} \int \frac{(u-1) + 3}{u^2 + 4} \, du$$

$$= \frac{1}{9} \int \frac{u}{u^2 + 4} \, du + \frac{1}{9} \int \frac{2}{u^2 + 2^2} \, du = \frac{1}{9} \cdot \frac{1}{2} \ln(u^2 + 4) + \frac{2}{9} \cdot \frac{1}{2} \tan^{-1}\left(\frac{1}{2} u \right) + C$$

$$= \tfrac{1}{18} \ln(9x^2 + 6x + 5) + \tfrac{1}{9} \tan^{-1}\left[\tfrac{1}{2}(3x+1) \right] + C$$

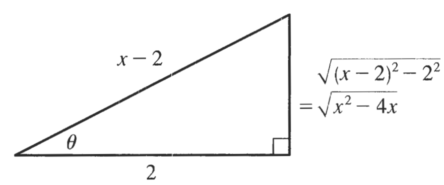
20. $$\int \tan^5 \theta \, \sec^3 \theta \, d\theta = \int \tan^4 \theta \, \sec^2 \theta \, \sec \theta \, \tan \theta \, d\theta = \int (\sec^2 \theta - 1)^2 \sec^2 \theta \, \sec \theta \, \tan \theta \, d\theta \qquad \begin{bmatrix} u = \sec \theta, \\ du = \sec \theta \, \tan \theta \, d\theta \end{bmatrix}$$

$$= \int (u^2 - 1)^2 u^2 \, du = \int (u^6 - 2u^4 + u^2) \, du$$

$$= \tfrac{1}{7} u^7 - \tfrac{2}{5} u^5 + \tfrac{1}{3} u^3 + C = \tfrac{1}{7} \sec^7 \theta - \tfrac{2}{5} \sec^5 \theta + \tfrac{1}{3} \sec^3 \theta + C$$

21. $$\int \frac{dx}{\sqrt{x^2 - 4x}} = \int \frac{dx}{\sqrt{(x^2 - 4x + 4) - 4}} = \int \frac{dx}{\sqrt{(x-2)^2 - 2^2}}$$

$$= \int \frac{2 \sec \theta \, \tan \theta \, d\theta}{2 \tan \theta} \qquad \begin{bmatrix} x - 2 = 2 \sec \theta, \\ dx = 2 \sec \theta \, \tan \theta \, d\theta \end{bmatrix}$$



$$= \int \sec \theta \, d\theta = \ln|\sec \theta + \tan \theta| + C_1$$

$$= \ln \left| \frac{x-2}{2} + \frac{\sqrt{x^2 - 4x}}{2} \right| + C_1$$

$$= \ln \left| x - 2 + \sqrt{x^2 - 4x} \right| + C, \text{ where } C = C_1 - \ln 2$$

22. Let $x = \sqrt{t}$, so that $x^2 = t$ and $2x \, dx = dt$. Then

$$\int t e^{\sqrt{t}} \, dt = \int x^2 e^x \, (2x \, dx) = \int 2x^3 e^x \, dx \qquad \begin{bmatrix} u_1 = 2x^3, & dv_1 = e^x \, dx, \\ du_1 = 6x^2 \, dx & v_1 = e^x \end{bmatrix}$$

$$= 2x^3 e^x - \int 6x^2 e^x \, dx \qquad \begin{bmatrix} u_2 = 6x^2, & dv_2 = e^x \, dx, \\ du_2 = 12x \, dx & v_2 = e^x \end{bmatrix}$$

$$= 2x^3 e^x - \left(6x^2 e^x - \int 12x e^x \, dx \right) \qquad \begin{bmatrix} u_3 = 12x, & dv_3 = e^x \, dx, \\ du_3 = 12 \, dx & v_3 = e^x \end{bmatrix}$$

$$= 2x^3 e^x - 6x^2 e^x + \left(12x e^x - \int 12 e^x \, dx \right) = 2x^3 e^x - 6x^2 e^x + 12x e^x - 12 e^x + C$$

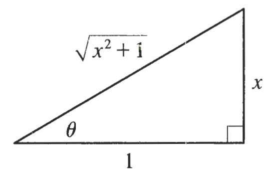
$$= 2e^x (x^3 - 3x^2 + 6x - 6) + C = 2e^{\sqrt{t}} \left(t \sqrt{t} - 3t + 6 \sqrt{t} - 6 \right) + C$$

23. Let $x = \tan\theta$, so that $dx = \sec^2\theta\, d\theta$. Then

$$\int \frac{dx}{x\sqrt{x^2+1}} = \int \frac{\sec^2\theta\, d\theta}{\tan\theta\,\sec\theta} = \int \frac{\sec\theta}{\tan\theta}\, d\theta$$

$$= \int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+1}}{x} - \frac{1}{x}\right| + C = \ln\left|\frac{\sqrt{x^2+1}-1}{x}\right| + C$$



24. Let $u = \cos x$, $dv = e^x\, dx \;\Rightarrow\; du = -\sin x\, dx$, $v = e^x$: $(*)\ I = \int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx$.

To integrate $\int e^x \sin x\, dx$, let $U = \sin x$, $dV = e^x\, dx \;\Rightarrow\; dU = \cos x\, dx$, $V = e^x$. Then

$\int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx = e^x \sin x - I$. By substitution in $(*)$, $I = e^x \cos x + e^x \sin x - I \;\Rightarrow$

$2I = e^x(\cos x + \sin x) \;\Rightarrow\; I = \frac{1}{2}e^x(\cos x + \sin x) + C$.

25. $\dfrac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2} \;\Rightarrow\; 3x^3 - x^2 + 6x - 4 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$.

Equating the coefficients gives $A + C = 3$, $B + D = -1$, $2A + C = 6$, and $2B + D = -4 \;\Rightarrow$

$A = 3$, $C = 0$, $B = -3$, and $D = 2$. Now

$$\int \frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)}\, dx = 3\int \frac{x-1}{x^2+1}\, dx + 2\int \frac{dx}{x^2+2} = \frac{3}{2}\ln(x^2+1) - 3\tan^{-1}x + \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C.$$

26. $\int x\sin x\,\cos x\, dx = \int \frac{1}{2}x\sin 2x\, dx \quad \begin{bmatrix} u = \frac{1}{2}x, & dv = \sin 2x\, dx, \\ du = \frac{1}{2}\, dx & v = -\frac{1}{2}\cos 2x \end{bmatrix}$

$$= -\tfrac{1}{4}x\cos 2x + \int \tfrac{1}{4}\cos 2x\, dx = -\tfrac{1}{4}x\cos 2x + \tfrac{1}{8}\sin 2x + C$$

27. $\int_0^{\pi/2} \cos^3 x\, \sin 2x\, dx = \int_0^{\pi/2} \cos^3 x\,(2\sin x\,\cos x)\, dx = \int_0^{\pi/2} 2\cos^4 x\, \sin x\, dx = \left[-\frac{2}{5}\cos^5 x\right]_0^{\pi/2} = \frac{2}{5}$

28. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2 du \;\Rightarrow$

$$\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\, dx = \int \frac{u+1}{u-1}\, 3u^2 du = 3\int \left(u^2 + 2u + 2 + \frac{2}{u-1}\right) du$$

$$= u^3 + 3u^2 + 6u + 6\ln|u-1| + C = x + 3x^{2/3} + 6\sqrt[3]{x} + 6\ln|\sqrt[3]{x} - 1| + C$$

29. The product of an odd function and an even function is an odd function, so $f(x) = x^5 \sec x$ is an odd function.

By Theorem 5.5.7(b), $\int_{-1}^{1} x^5 \sec x\, dx = 0$.

30. Let $u = e^{-x}$, $du = -e^{-x}\, dx$. Then

$$\int \frac{dx}{e^x\sqrt{1 - e^{-2x}}} = \int \frac{e^{-x}\, dx}{\sqrt{1 - (e^{-x})^2}} = \int \frac{-du}{\sqrt{1-u^2}} = -\sin^{-1}u + C = -\sin^{-1}(e^{-x}) + C.$$

31. Let $u = \sqrt{e^x - 1}$. Then $u^2 = e^x - 1$ and $2u\, du = e^x\, dx$. Also, $e^x + 8 = u^2 + 9$. Thus,

$$\int_0^{\ln 10} \frac{e^x\sqrt{e^x-1}}{e^x+8}\, dx = \int_0^3 \frac{u \cdot 2u\, du}{u^2+9} = 2\int_0^3 \frac{u^2}{u^2+9}\, du = 2\int_0^3 \left(1 - \frac{9}{u^2+9}\right) du$$

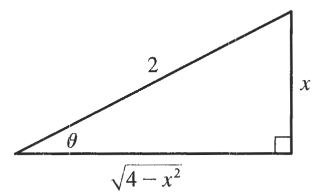
$$= 2\left[u - \frac{9}{3}\tan^{-1}\left(\frac{u}{3}\right)\right]_0^3 = 2[(3 - 3\tan^{-1}1) - 0] = 2\left(3 - 3\cdot\frac{\pi}{4}\right) = 6 - \frac{3\pi}{2}$$

32. $\displaystyle\int_0^{\pi/4} \frac{x\sin x}{\cos^3 x}\,dx = \int_0^{\pi/4} x\tan x\,\sec^2 x\,dx \qquad \begin{bmatrix} u = x, & dv = \tan x\,\sec^2 x\,dx, \\ du = dx & v = \frac{1}{2}\tan^2 x \end{bmatrix}$

$$= \left[\frac{x}{2}\tan^2 x\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4} \tan^2 x\,dx = \frac{\pi}{8}\cdot 1^2 - 0 - \frac{1}{2}\int_0^{\pi/4}(\sec^2 x - 1)\,dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\Big[\tan x - x\Big]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2}$$

33. Let $x = 2\sin\theta \;\Rightarrow\; \left(4 - x^2\right)^{3/2} = (2\cos\theta)^3$, $dx = 2\cos\theta\,d\theta$, so



$$\int \frac{x^2}{\left(4 - x^2\right)^{3/2}}\,dx = \int \frac{4\sin^2\theta}{8\cos^3\theta}\,2\cos\theta\,d\theta = \int \tan^2\theta\,d\theta = \int\left(\sec^2\theta - 1\right)d\theta$$

$$= \tan\theta - \theta + C = \frac{x}{\sqrt{4 - x^2}} - \sin^{-1}\!\left(\frac{x}{2}\right) + C$$

34. Integrate by parts twice, first with $u = (\arcsin x)^2$, $dv = dx$:

$$I = \int(\arcsin x)^2\,dx = x(\arcsin x)^2 - \int 2x\arcsin x\left(\frac{dx}{\sqrt{1 - x^2}}\right)$$

Now let $U = \arcsin x$, $dV = \dfrac{x}{\sqrt{1 - x^2}}\,dx \;\Rightarrow\; dU = \dfrac{1}{\sqrt{1 - x^2}}\,dx$, $V = -\sqrt{1 - x^2}$. So

$$I = x(\arcsin x)^2 - 2\left[\arcsin x\left(-\sqrt{1 - x^2}\right) + \textstyle\int dx\right] = x(\arcsin x)^2 + 2\sqrt{1 - x^2}\arcsin x - 2x + C$$

35. $\displaystyle\int \frac{1}{\sqrt{x + x^{3/2}}}\,dx = \int \frac{dx}{\sqrt{x\left(1 + \sqrt{x}\right)}} = \int \frac{dx}{\sqrt{x}\sqrt{1 + \sqrt{x}}} \quad \begin{bmatrix} u = 1 + \sqrt{x}, \\ du = \dfrac{dx}{2\sqrt{x}} \end{bmatrix} \quad = \int \frac{2\,du}{\sqrt{u}} = \int 2u^{-1/2}\,du$

$$= 4\sqrt{u} + C = 4\sqrt{1 + \sqrt{x}} + C$$

36. $\displaystyle\int \frac{1 - \tan\theta}{1 + \tan\theta}\,d\theta = \int \frac{\dfrac{\cos\theta}{\cos\theta} - \dfrac{\sin\theta}{\cos\theta}}{\dfrac{\cos\theta}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}}\,d\theta = \int \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\,d\theta = \ln|\cos\theta + \sin\theta| + C$

37. $\int(\cos x + \sin x)^2\cos 2x\,dx = \int\left(\cos^2 x + 2\sin x\cos x + \sin^2 x\right)\cos 2x\,dx = \int(1 + \sin 2x)\cos 2x\,dx$

$$= \textstyle\int\cos 2x\,dx + \frac{1}{2}\int\sin 4x\,dx = \frac{1}{2}\sin 2x - \frac{1}{8}\cos 4x + C$$

Or: $\int(\cos x + \sin x)^2\cos 2x\,dx = \int(\cos x + \sin x)^2\left(\cos^2 x - \sin^2 x\right)dx$

$$= \textstyle\int(\cos x + \sin x)^3(\cos x - \sin x)\,dx = \frac{1}{4}(\cos x + \sin x)^4 + C_1$$

38. Let $u = x + 2$, so that $du = dx$ and $x = u - 2$. Then

$$\int \frac{x^2}{(x + 2)^3}\,dx = \int \frac{(u - 2)^2}{u^3}\,du = \int \frac{u^2 - 4u + 4}{u^3}\,du = \int\left(\frac{1}{u} - 4u^{-2} + 4u^{-3}\right)du$$

$$= \ln|u| + 4u^{-1} - 2u^{-2} + C = \ln|x + 2| + \frac{4}{x + 2} - \frac{2}{(x + 2)^2} + C$$

39. We'll integrate $I = \int \frac{xe^{2x}}{(1+2x)^2}\,dx$ by parts with $u = xe^{2x}$ and $dv = \frac{dx}{(1+2x)^2}$. Then $du = (x \cdot 2e^{2x} + e^{2x} \cdot 1)\,dx$ and $v = -\frac{1}{2} \cdot \frac{1}{1+2x}$, so

$$I = -\frac{1}{2} \cdot \frac{xe^{2x}}{1+2x} - \int \left[-\frac{1}{2} \cdot \frac{e^{2x}(2x+1)}{1+2x} \right] dx = -\frac{xe^{2x}}{4x+2} + \frac{1}{2} \cdot \frac{1}{2} e^{2x} + C = e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right) + C$$

Thus, $\int_0^{1/2} \frac{xe^{2x}}{(1+2x)^2}\,dx = \left[e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right)\right]_0^{1/2} = e\left(\frac{1}{4} - \frac{1}{8}\right) - 1\left(\frac{1}{4} - 0\right) = \frac{1}{8}e - \frac{1}{4}$.

40. $\int_{\pi/4}^{\pi/3} \frac{\sqrt{\tan\theta}}{\sin 2\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\sqrt{\frac{\sin\theta}{\cos\theta}}}{2\sin\theta\cos\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{2}(\sin\theta)^{-1/2}(\cos\theta)^{-3/2}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{2}\left(\frac{\sin\theta}{\cos\theta}\right)^{-1/2}(\cos\theta)^{-2}d\theta$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta\,d\theta = \left[\sqrt{\tan\theta}\right]_{\pi/4}^{\pi/3} = \sqrt{\sqrt{3}} - \sqrt{1} = \sqrt[4]{3} - 1$$

41. $\int_1^\infty \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{2}(2x+1)^{-3}\,2\,dx = \lim_{t\to\infty}\left[-\frac{1}{4(2x+1)^2}\right]_1^t$

$$= -\frac{1}{4}\lim_{t\to\infty}\left[\frac{1}{(2t+1)^2} - \frac{1}{9}\right] = -\frac{1}{4}\left(0 - \frac{1}{9}\right) = \frac{1}{36}$$

42. $\int_1^\infty \frac{\ln x}{x^4}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^4}\,dx \quad \left[\begin{matrix} u = \ln x, & dv = dx/x^4, \\ du = dx/x & v = -1/(3x^3) \end{matrix}\right]$

$$= \lim_{t\to\infty}\left[-\frac{\ln x}{3x^3}\right]_1^t + \int_1^t \frac{1}{3x^4}\,dx = \lim_{t\to\infty}\left(-\frac{\ln t}{3t^3} + 0 + \left[\frac{-1}{9x^3}\right]_1^t\right) \stackrel{\text{H}}{=} \lim_{t\to\infty}\left(-\frac{1}{9t^3} + \left[\frac{-1}{9t^3} + \frac{1}{9}\right]\right)$$

$$= 0 + 0 + \tfrac{1}{9} = \tfrac{1}{9}$$

43. $\int \frac{dx}{x\ln x} \quad \left[\begin{matrix} u = \ln x, \\ du = dx/x \end{matrix}\right] \quad = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$, so

$\int_2^\infty \frac{dx}{x\ln x} = \lim_{t\to\infty}\int_2^t \frac{dx}{x\ln x} = \lim_{t\to\infty}\left[\ln|\ln x|\right]_2^t = \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right] = \infty$, so the integral is divergent.

44. Let $u = \sqrt{y-2}$. Then $y = u^2 + 2$ and $dy = 2u\,du$, so

$$\int \frac{y\,dy}{\sqrt{y-2}} = \int \frac{(u^2+2)2u\,du}{u} = 2\int (u^2+2)\,du = 2\left[\tfrac{1}{3}u^3 + 2u\right] + C$$

Thus, $\int_2^6 \frac{y\,dy}{\sqrt{y-2}} = \lim_{t\to 2^+}\int_t^6 \frac{y\,dy}{\sqrt{y-2}} = \lim_{t\to 2^+}\left[\tfrac{2}{3}(y-2)^{3/2} + 4\sqrt{y-2}\right]_t^6$

$$= \lim_{t\to 2^+}\left[\tfrac{16}{3} + 8 - \tfrac{2}{3}(t-2)^{3/2} - 4\sqrt{t-2}\right] = \tfrac{40}{3}.$$

45. $\displaystyle\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^4 \frac{\ln x}{\sqrt{x}}\,dx \overset{*}{=} \lim_{t\to 0^+}\left[2\sqrt{x}\,\ln x - 4\sqrt{x}\right]_t^4$

$\displaystyle = \lim_{t\to 0^+}\left[(2\cdot 2\ln 4 - 4\cdot 2) - \left(2\sqrt{t}\,\ln t - 4\sqrt{t}\right)\right] \overset{**}{=} (4\ln 4 - 8) - (0-0) = 4\ln 4 - 8$

(∗) Let $u = \ln x$, $dv = \dfrac{1}{\sqrt{x}}\,dx \;\Rightarrow\; du = \dfrac{1}{x}\,dx$, $v = 2\sqrt{x}$. Then

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\,\ln x - 2\int\frac{dx}{\sqrt{x}} = 2\sqrt{x}\,\ln x - 4\sqrt{x} + C$$

(∗∗) $\displaystyle\lim_{t\to 0^+}\left(2\sqrt{t}\,\ln t\right) = \lim_{t\to 0^+}\frac{2\ln t}{t^{-1/2}} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{2/t}{-\frac{1}{2}t^{-3/2}} = \lim_{t\to 0^+}\left(-4\sqrt{t}\right) = 0$

46. Note that $f(x) = 1/(2-3x)$ has an infinite discontinuity at $x = \frac{2}{3}$. Now

$$\int_0^{2/3}\frac{1}{2-3x}\,dx = \lim_{t\to(2/3)^-}\int_0^t \frac{1}{2-3x}\,dx = \lim_{t\to(2/3)^-}\left[-\tfrac{1}{3}\ln|2-3x|\right]_0^t = -\tfrac{1}{3}\lim_{t\to(2/3)^-}\left[\ln|2-3t| - \ln 2\right] = \infty$$

Since $\displaystyle\int_0^{2/3}\frac{1}{2-3x}\,dx$ diverges, so does $\displaystyle\int_0^1\frac{1}{2-3x}\,dx$.

47. $\displaystyle\int_0^1\frac{x-1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1\left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right)dx = \lim_{t\to 0^+}\int_t^1\left(x^{1/2} - x^{-1/2}\right)dx = \lim_{t\to 0^+}\left[\tfrac{2}{3}x^{3/2} - 2x^{1/2}\right]_t^1$

$\displaystyle = \lim_{t\to 0^+}\left[\left(\tfrac{2}{3} - 2\right) - \left(\tfrac{2}{3}t^{3/2} - 2t^{1/2}\right)\right] = -\tfrac{4}{3} - 0 = -\tfrac{4}{3}$

48. $\displaystyle I = \int_{-1}^1\frac{dx}{x^2-2x} = \int_{-1}^1\frac{dx}{x(x-2)} = \int_{-1}^0\frac{dx}{x(x-2)} + \int_0^1\frac{dx}{x(x-2)} = I_1 + I_2$. Now

$\dfrac{1}{x(x-2)} = \dfrac{A}{x} + \dfrac{B}{x-2} \;\Rightarrow\; 1 = A(x-2) + Bx$. Set $x = 2$ to get $1 = 2B$, so $B = \frac{1}{2}$. Set $x = 0$ to get $1 = -2A$, $A = -\frac{1}{2}$. Thus,

$$I_2 = \lim_{t\to 0^+}\int_t^1\left(\frac{-\frac{1}{2}}{x} + \frac{\frac{1}{2}}{x-2}\right)dx = \lim_{t\to 0^+}\left[-\tfrac{1}{2}\ln|x| + \tfrac{1}{2}\ln|x-2|\right]_t^1 = \lim_{t\to 0^+}\left[(0+0) - \left(-\tfrac{1}{2}\ln t + \tfrac{1}{2}\ln|t-2|\right)\right]$$

$\displaystyle = -\tfrac{1}{2}\ln 2 + \tfrac{1}{2}\lim_{t\to 0^+}\ln t = -\infty.$

Since I_2 diverges, I is divergent.

49. Let $u = 2x+1$. Then

$$\int_{-\infty}^{\infty}\frac{dx}{4x^2+4x+5} = \int_{-\infty}^{\infty}\frac{\frac{1}{2}\,du}{u^2+4} = \frac{1}{2}\int_{-\infty}^0\frac{du}{u^2+4} + \frac{1}{2}\int_0^{\infty}\frac{du}{u^2+4}$$

$$= \tfrac{1}{2}\lim_{t\to-\infty}\left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right)\right]_t^0 + \tfrac{1}{2}\lim_{t\to\infty}\left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right)\right]_0^t = \tfrac{1}{4}\left[0 - \left(-\tfrac{\pi}{2}\right)\right] + \tfrac{1}{4}\left[\tfrac{\pi}{2} - 0\right] = \tfrac{\pi}{4}.$$

50. $\displaystyle\int_1^\infty \frac{\tan^{-1} x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\tan^{-1} x}{x^2}\,dx$. Integrate by parts:

$$\int \frac{\tan^{-1} x}{x^2}\,dx = \frac{-\tan^{-1} x}{x} + \int \frac{1}{x}\frac{dx}{1+x^2} = \frac{-\tan^{-1} x}{x} + \int\left[\frac{1}{x} - \frac{x}{x^2+1}\right]dx$$

$$= \frac{-\tan^{-1} x}{x} + \ln|x| - \tfrac{1}{2}\ln(x^2+1) + C = \frac{-\tan^{-1} x}{x} + \frac{1}{2}\ln\frac{x^2}{x^2+1} + C$$

Thus,

$$\int_1^\infty \frac{\tan^{-1} x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\tan^{-1} x}{x} + \frac{1}{2}\ln\frac{x^2}{x^2+1}\right]_1^t = \lim_{t\to\infty}\left[-\frac{\tan^{-1} t}{t} + \frac{1}{2}\ln\frac{t^2}{t^2+1} + \frac{\pi}{4} - \frac{1}{2}\ln\frac{1}{2}\right]$$

$$= 0 + \tfrac{1}{2}\ln 1 + \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2 = \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2$$

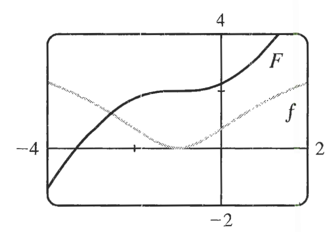
51. We first make the substitution $t = x+1$, so $\ln(x^2+2x+2) = \ln\left[(x+1)^2+1\right] = \ln(t^2+1)$. Then we use parts with $u = \ln(t^2+1)$, $dv = dt$:

$$\int \ln(t^2+1)\,dt = t\,\ln(t^2+1) - \int \frac{t(2t)\,dt}{t^2+1} = t\,\ln(t^2+1) - 2\int\frac{t^2\,dt}{t^2+1} = t\,\ln(t^2+1) - 2\int\left(1 - \frac{1}{t^2+1}\right)dt$$

$$= t\,\ln(t^2+1) - 2t + 2\arctan t + C$$

$$= (x+1)\,\ln(x^2+2x+2) - 2x + 2\arctan(x+1) + K, \text{ where } K = C - 2$$

[Alternatively, we could have integrated by parts immediately with $u = \ln(x^2+2x+2)$.] Notice from the graph that $f = 0$ where F has a horizontal tangent. Also, F is always increasing, and $f \geq 0$.

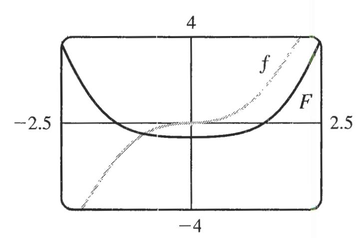


52. Let $u = x^2+1$. Then $x^2 = u-1$ and $x\,dx = \frac{1}{2}\,du$, so

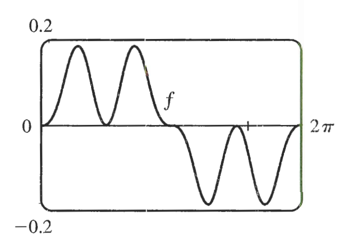
$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{(u-1)}{\sqrt{u}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int\left(u^{1/2} - u^{-1/2}\right)du$$

$$= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C = \tfrac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2} + C$$

$$= \tfrac{1}{3}(x^2+1)^{1/2}\left[(x^2+1) - 3\right] + C = \tfrac{1}{3}\sqrt{x^2+1}\,(x^2-2) + C$$



53. From the graph, it seems as though $\int_0^{2\pi}\cos^2 x\,\sin^3 x\,dx$ is equal to 0. To evaluate the integral, we write the integral as $I = \int_0^{2\pi}\cos^2 x\,(1-\cos^2 x)\,\sin x\,dx$ and let $u = \cos x \;\Rightarrow\; du = -\sin x\,dx$. Thus, $I = \int_1^1 u^2(1-u^2)(-du) = 0$.

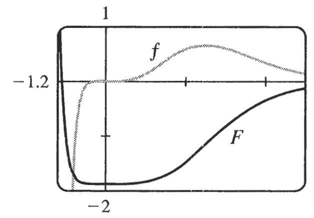


54. (a) To evaluate $\int x^5 e^{-2x}\,dx$ by hand, we would integrate by parts repeatedly, always taking $dv = e^{-2x}$ and starting with $u = x^5$. Each time we would reduce the degree of the x-factor by 1.

(b) To evaluate the integral using tables, we would use Formula 97 (which is proved using integration by parts) until the exponent of x was reduced to 1, and then we would use Formula 96.

(c) $\int x^5 e^{-2x}\,dx = -\frac{1}{8}e^{-2x}(4x^5 + 10x^4 + 20x^3 + 30x^2 + 30x + 15) + C$

(d)



55. $\int \sqrt{4x^2 - 4x - 3}\,dx = \int \sqrt{(2x-1)^2 - 4}\,dx \quad \left[\begin{array}{l} u = 2x - 1, \\ du = 2\,dx \end{array}\right] \quad = \int \sqrt{u^2 - 2^2}\left(\frac{1}{2}\,du\right)$

$$\overset{39}{=} \frac{1}{2}\left(\frac{u}{2}\sqrt{u^2 - 2^2} - \frac{2^2}{2}\ln\left|u + \sqrt{u^2 - 2^2}\right|\right) + C = \tfrac{1}{4}u\sqrt{u^2 - 4} - \ln\left|u + \sqrt{u^2 - 4}\right| + C$$

$$= \tfrac{1}{4}(2x - 1)\sqrt{4x^2 - 4x - 3} - \ln\left|2x - 1 + \sqrt{4x^2 - 4x - 3}\right| + C$$

56. $\int \csc^5 t\,dt \overset{78}{=} -\frac{1}{4}\cot t\,\csc^3 t + \frac{3}{4}\int \csc^3 t\,dt \overset{72}{=} -\frac{1}{4}\cot t\,\csc^3 t + \frac{3}{4}\left[-\frac{1}{2}\csc t\,\cot t + \frac{1}{2}\ln|\csc t - \cot t|\right] + C$

$$= -\tfrac{1}{4}\cot t\,\csc^3 t - \tfrac{3}{8}\csc t\,\cot t + \tfrac{3}{8}\ln|\csc t - \cot t| + C$$

57. Let $u = \sin x$, so that $du = \cos x\,dx$. Then

$$\int \cos x\sqrt{4 + \sin^2 x}\,dx = \int \sqrt{2^2 + u^2}\,du \overset{21}{=} \frac{u}{2}\sqrt{2^2 + u^2} + \frac{2^2}{2}\ln\left(u + \sqrt{2^2 + u^2}\right) + C$$

$$= \tfrac{1}{2}\sin x\sqrt{4 + \sin^2 x} + 2\ln\left(\sin x + \sqrt{4 + \sin^2 x}\right) + C$$

58. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cot x\,dx}{\sqrt{1 + 2\sin x}} = \int \frac{du}{u\sqrt{1 + 2u}} \overset{\substack{57\text{ with}\\ a=1,\,b=2}}{=} \ln\left|\frac{\sqrt{1 + 2u} - 1}{\sqrt{1 + 2u} + 1}\right| + C = \ln\left|\frac{\sqrt{1 + 2\sin x} - 1}{\sqrt{1 + 2\sin x} + 1}\right| + C$$

59. (a) $\dfrac{d}{du}\left[-\dfrac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\left(\dfrac{u}{a}\right) + C\right] = \dfrac{1}{u^2}\sqrt{a^2 - u^2} + \dfrac{1}{\sqrt{a^2 - u^2}} - \dfrac{1}{\sqrt{1 - u^2/a^2}} \cdot \dfrac{1}{a}$

$$= \left(a^2 - u^2\right)^{-1/2}\left[\frac{1}{u^2}\left(a^2 - u^2\right) + 1 - 1\right] = \frac{\sqrt{a^2 - u^2}}{u^2}$$

(b) Let $u = a\sin\theta \;\Rightarrow\; du = a\cos\theta\,d\theta,\; a^2 - u^2 = a^2\left(1 - \sin^2\theta\right) = a^2\cos^2\theta$.

$$\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = \int \frac{a^2\cos^2\theta}{a^2\sin^2\theta}\,d\theta = \int \frac{1 - \sin^2\theta}{\sin^2\theta}\,d\theta = \int \left(\csc^2\theta - 1\right)d\theta = -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{a^2 - u^2}}{u} - \sin^{-1}\left(\frac{u}{a}\right) + C$$

60. Work backward, and use integration by parts with $U = u^{-(n-1)}$ and $dV = (a+bu)^{-1/2}\,du \Rightarrow$

$dU = \dfrac{-(n-1)\,du}{u^n}$ and $V = \dfrac{2}{b}\sqrt{a+bu}$, to get

$$\int \frac{du}{u^{n-1}\sqrt{a+bu}} = \int U\,dV = UV - \int V\,dU = \frac{2\sqrt{a+bu}}{bu^{n-1}} + \frac{2(n-1)}{b}\int \frac{\sqrt{a+bu}}{u^n}\,du$$

$$= \frac{2\sqrt{a+bu}}{bu^{n-1}} + \frac{2(n-1)}{b}\int \frac{a+bu}{u^n\sqrt{a+bu}}\,du$$

$$= \frac{2\sqrt{a+bu}}{bu^{n-1}} + 2(n-1)\int \frac{du}{u^{n-1}\sqrt{a+bu}} + \frac{2a(n-1)}{b}\int \frac{du}{u^n\sqrt{a+bu}}$$

Rearranging the equation gives $\dfrac{2a(n-1)}{b}\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{2\sqrt{a+bu}}{bu^{n-1}} - (2n-3)\int \frac{du}{u^{n-1}\sqrt{a+bu}} \Rightarrow$

$$\int \frac{du}{u^n\sqrt{a+bu}} = \frac{-\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

61. For $n \geq 0$, $\int_0^\infty x^n\,dx = \lim\limits_{t\to\infty}\left[x^{n+1}/(n+1)\right]_0^t = \infty$. For $n < 0$, $\int_0^\infty x^n\,dx = \int_0^1 x^n\,dx + \int_1^\infty x^n\,dx$. Both integrals are improper. By (7.8.2), the second integral diverges if $-1 \leq n < 0$. By Exercise 7.8.57, the first integral diverges if $n \leq -1$. Thus, $\int_0^\infty x^n\,dx$ is divergent for all values of n.

62. $I = \displaystyle\int_0^\infty e^{ax}\cos x\,dx = \lim_{t\to\infty}\int_0^t e^{ax}\cos x\,dx \overset{\substack{99 \text{ with}\\ b=1}}{=} \lim_{t\to\infty}\left[\frac{e^{ax}}{a^2+1}(a\cos x + \sin x)\right]_0^t$

$$= \lim_{t\to\infty}\left[\frac{e^{at}}{a^2+1}(a\cos t + \sin t) - \frac{1}{a^2+1}(a)\right] = \frac{1}{a^2+1}\lim_{t\to\infty}\left[e^{at}(a\cos t + \sin t) - a\right].$$

For $a \geq 0$, the limit does not exist due to oscillation. For $a < 0$, $\lim\limits_{t\to\infty}\left[e^{at}(a\cos t + \sin t)\right] = 0$ by the Squeeze Theorem, because $\left|e^{at}(a\cos t + \sin t)\right| \leq e^{at}(|a|+1)$, so $I = \dfrac{1}{a^2+1}(-a) = -\dfrac{a}{a^2+1}$.

63. $f(x) = \dfrac{1}{\ln x}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{4-2}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5\cdot 2}\{f(2) + 2[f(2.2) + f(2.4) + \cdots + f(3.8)] + f(4)\} \approx 1.925444$

(b) $M_{10} = \frac{1}{5}[f(2.1) + f(2.3) + f(2.5) + \cdots + f(3.9)] \approx 1.920915$

(c) $S_{10} = \frac{1}{5\cdot 3}[f(2) + 4f(2.2) + 2f(2.4) + \cdots + 2f(3.6) + 4f(3.8) + f(4)] \approx 1.922470$

64. $f(x) = \sqrt{x}\cos x$, $\Delta x = \frac{b-a}{n} = \frac{4-1}{10} = \frac{3}{10}$

(a) $T_{10} = \frac{3}{10\cdot 2}\{f(1) + 2[f(1.3) + f(1.6) + \cdots + f(3.7)] + f(4)\} \approx -2.835151$

(b) $M_{10} = \frac{3}{10}[f(1.15) + f(1.45) + f(1.75) + \cdots + f(3.85)] \approx -2.856809$

(c) $S_{10} = \frac{3}{10\cdot 3}[f(1) + 4f(1.3) + 2f(1.6) + \cdots + 2f(3.4) + 4f(3.7) + f(4)] \approx -2.849672$

65. $f(x) = \dfrac{1}{\ln x} \;\Rightarrow\; f'(x) = -\dfrac{1}{x(\ln x)^2} \;\Rightarrow\; f''(x) = \dfrac{2+\ln x}{x^2(\ln x)^3} = \dfrac{2}{x^2(\ln x)^3} + \dfrac{1}{x^2(\ln x)^2}$. Note that each term of $f''(x)$ decreases on $[2,4]$, so we'll take $K = f''(2) \approx 2.022$. $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \approx \dfrac{2.022(4-2)^3}{12(10)^2} = 0.01348$ and $|E_M| \le \dfrac{K(b-a)^3}{24n^2} = 0.00674$. $|E_T| \le 0.00001 \;\Leftrightarrow\; \dfrac{2.022(8)}{12n^2} \le \dfrac{1}{10^5} \;\Leftrightarrow\; n^2 \ge \dfrac{10^5(2.022)(8)}{12} \;\Rightarrow\; n \ge 367.2$.

Take $n = 368$ for T_n. $|E_M| \le 0.00001 \;\Leftrightarrow\; n^2 \ge \dfrac{10^5(2.022)(8)}{24} \;\Rightarrow\; n \ge 259.6$. Take $n = 260$ for M_n.

66. $\displaystyle\int_1^4 \frac{e^x}{x}\,dx \approx S_6 = \frac{(4-1)/6}{3}\,[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)] \approx 17.739438$

67. $\Delta t = \left(\frac{10}{60} - 0\right)/\,10 = \frac{1}{60}$.

$$\begin{aligned}\text{Distance traveled} &= \textstyle\int_0^{10} v\,dt \approx S_{10}\\ &= \tfrac{1}{60\cdot 3}[40 + 4(42) + 2(45) + 4(49) + 2(52) + 4(54) + 2(56) + 4(57) + 2(57) + 4(55) + 56]\\ &= \tfrac{1}{180}(1544) = 8.5\overline{7}\text{ mi}\end{aligned}$$

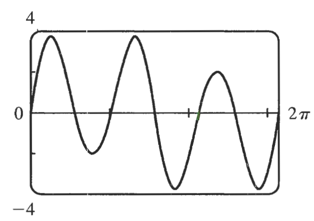
68. We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$:

$$\begin{aligned}\text{Increase in bee population} &= \textstyle\int_0^{24} r(t)\,dt \approx S_6\\ &= \tfrac{4}{3}[r(0) + 4r(4) + 2r(8) + 4r(12) + 2r(16) + 4r(20) + r(24)]\\ &= \tfrac{4}{3}[0 + 4(300) + 2(3000) + 4(11{,}000) + 2(4000) + 4(400) + 0]\\ &= \tfrac{4}{3}(60{,}800) \approx 81{,}067\text{ bees}\end{aligned}$$

69. (a) $f(x) = \sin(\sin x)$. A CAS gives

$$\begin{aligned}f^{(4)}(x) = {}&\sin(\sin x)[\cos^4 x + 7\cos^2 x - 3]\\ &+ \cos(\sin x)\,[6\cos^2 x \sin x + \sin x]\end{aligned}$$

From the graph, we see that $\left|f^{(4)}(x)\right| < 3.8$ for $x \in [0,\pi]$.



(b) We use Simpson's Rule with $f(x) = \sin(\sin x)$ and $\Delta x = \frac{\pi}{10}$:

$$\textstyle\int_0^\pi f(x)\,dx \approx \frac{\pi}{10\cdot 3}\left[f(0) + 4f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + \cdots + 4f\left(\frac{9\pi}{10}\right) + f(\pi)\right] \approx 1.786721$$

From part (a), we know that $\left|f^{(4)}(x)\right| < 3.8$ on $[0,\pi]$, so we use Theorem 7.7.4 with $K = 3.8$, and estimate the error as $|E_S| \le \dfrac{3.8(\pi - 0)^5}{180(10)^4} \approx 0.000646$.

(c) If we want the error to be less than 0.00001, we must have $|E_S| \le \dfrac{3.8\pi^5}{180n^4} \le 0.00001$, so $n^4 \ge \dfrac{3.8\pi^5}{180(0.00001)} \approx 646{,}041.6 \;\Rightarrow\; n \ge 28.35$. Since n must be even for Simpson's Rule, we must have $n \ge 30$ to ensure the desired accuracy.

70. With an x-axis in the normal position, at $x = 7$ we have $C = 2\pi r = 45 \Rightarrow r(7) = \frac{2\pi}{45}$.

Using Simpson's Rule with $n = 4$ and $\Delta x = 7$, we have

$V = \int_0^{28} \pi[r(x)]^2\,dx \approx S_4 = \frac{7}{3}\left[0 + 4\pi\left(\frac{45}{2\pi}\right)^2 + 2\pi\left(\frac{53}{2\pi}\right)^2 + 4\pi\left(\frac{45}{2\pi}\right)^2 + 0\right] = \frac{7}{3}\left(\frac{21{,}818}{4\pi}\right) \approx 4051\text{ cm}^3.$

71. $\dfrac{x^3}{x^5+2} \le \dfrac{x^3}{x^5} = \dfrac{1}{x^2}$ for x in $[1, \infty)$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by (7.8.2) with $p = 2 > 1$. Therefore, $\displaystyle\int_1^\infty \frac{x^3}{x^5+2}\,dx$ is convergent by the Comparison Theorem.

72. The line $y = 3$ intersects the hyperbola $y^2 - x^2 = 1$ at two points on its upper branch, namely $(-2\sqrt{2}, 3)$ and $(2\sqrt{2}, 3)$. The desired area is

$$A = \int_{-2\sqrt{2}}^{2\sqrt{2}} \left(3 - \sqrt{x^2+1}\right) dx = 2\int_0^{2\sqrt{2}} \left(3 - \sqrt{x^2+1}\right) dx \overset{21}{=} 2\left[3x - \tfrac{1}{2}x\sqrt{x^2+1} - \tfrac{1}{2}\ln\left(x + \sqrt{x^2+1}\right)\right]_0^{2\sqrt{2}}$$

$$= \left[6x - x\sqrt{x^2+1} - \ln\left(x + \sqrt{x^2+1}\right)\right]_0^{2\sqrt{2}} = 12\sqrt{2} - 2\sqrt{2}\cdot 3 - \ln(2\sqrt{2}+3) = 6\sqrt{2} - \ln(3 + 2\sqrt{2})$$

Another method: $A = 2\int_1^3 \sqrt{y^2-1}\,dy$ and use Formula 39.

73. For x in $\left[0, \frac{\pi}{2}\right]$, $0 \le \cos^2 x \le \cos x$. For x in $\left[\frac{\pi}{2}, \pi\right]$, $\cos x \le 0 \le \cos^2 x$. Thus,

$$\text{area} = \int_0^{\pi/2} (\cos x - \cos^2 x)\,dx + \int_{\pi/2}^{\pi} (\cos^2 x - \cos x)\,dx$$

$$= \left[\sin x - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\pi/2} + \left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x - \sin x\right]_{\pi/2}^{\pi} = \left[\left(1 - \tfrac{\pi}{4}\right) - 0\right] + \left[\tfrac{\pi}{2} - \left(\tfrac{\pi}{4} - 1\right)\right] = 2$$

74. The curves $y = \dfrac{1}{2 \pm \sqrt{x}}$ are defined for $x \ge 0$. For $x > 0$, $\dfrac{1}{2 - \sqrt{x}} > \dfrac{1}{2 + \sqrt{x}}$. Thus, the required area is

$$\int_0^1 \left(\frac{1}{2-\sqrt{x}} - \frac{1}{2+\sqrt{x}}\right) dx = \int_0^1 \left(\frac{1}{2-u} - \frac{1}{2+u}\right) 2u\,du \quad \left[u = \sqrt{x}\right] \quad = 2\int_0^1 \left(-\frac{u}{u-2} - \frac{u}{u+2}\right) du$$

$$= 2\int_0^1 \left(-1 - \frac{2}{u-2} - 1 + \frac{2}{u+2}\right) du = 2\left[2\ln\left|\frac{u+2}{u-2}\right| - 2u\right]_0^1 = 4\ln 3 - 4.$$

75. Using the formula for disks, the volume is

$$V = \int_0^{\pi/2} \pi\,[f(x)]^2\,dx = \pi\int_0^{\pi/2} (\cos^2 x)^2\,dx = \pi\int_0^{\pi/2} \left[\tfrac{1}{2}(1 + \cos 2x)\right]^2 dx$$

$$= \tfrac{\pi}{4}\int_0^{\pi/2} (1 + \cos^2 2x + 2\cos 2x)\,dx = \tfrac{\pi}{4}\int_0^{\pi/2} \left[1 + \tfrac{1}{2}(1 + \cos 4x) + 2\cos 2x\right] dx$$

$$= \tfrac{\pi}{4}\left[\tfrac{3}{2}x + \tfrac{1}{2}\left(\tfrac{1}{4}\sin 4x\right) + 2\left(\tfrac{1}{2}\sin 2x\right)\right]_0^{\pi/2} = \tfrac{\pi}{4}\left[\left(\tfrac{3\pi}{4} + \tfrac{1}{8}\cdot 0 + 0\right) - 0\right] = \tfrac{3\pi^2}{16}$$

76. Using the formula for cylindrical shells, the volume is

$$V = \int_0^{\pi/2} 2\pi x f(x)\,dx = 2\pi\int_0^{\pi/2} x\cos^2 x\,dx = 2\pi\int_0^{\pi/2} x\left[\tfrac{1}{2}(1 + \cos 2x)\right] dx = 2\left(\tfrac{1}{2}\right)\pi\int_0^{\pi/2} (x + x\cos 2x)\,dx$$

$$= \pi\left(\left[\tfrac{1}{2}x^2\right]_0^{\pi/2} + \left[x\left(\tfrac{1}{2}\sin 2x\right)\right]_0^{\pi/2} - \int_0^{\pi/2} \tfrac{1}{2}\sin 2x\,dx\right) \quad \left[\begin{array}{l}\text{parts with } u = x,\\ dv = \cos 2x\,dx\end{array}\right]$$

$$= \pi\left[\tfrac{1}{2}\left(\tfrac{\pi}{2}\right)^2 + 0 - \tfrac{1}{2}\left[-\tfrac{1}{2}\cos 2x\right]_0^{\pi/2}\right] = \tfrac{\pi^3}{8} + \tfrac{\pi}{4}(-1 - 1) = \tfrac{1}{8}(\pi^3 - 4\pi)$$

77. By the Fundamental Theorem of Calculus,

$\int_0^\infty f'(x)\,dx = \lim\limits_{t\to\infty} \int_0^t f'(x)\,dx = \lim\limits_{t\to\infty} [f(t) - f(0)] = \lim\limits_{t\to\infty} f(t) - f(0) = 0 - f(0) = -f(0).$

78. (a) $(\tan^{-1} x)_{\text{ave}} = \lim\limits_{t\to\infty} \dfrac{1}{t-0} \displaystyle\int_0^t \tan^{-1} x\,dx \overset{89}{=} \lim\limits_{t\to\infty} \left\{ \dfrac{1}{t} \left[x \tan^{-1} x - \tfrac{1}{2} \ln(1+x^2) \right]_0^t \right\}$

$$= \lim_{t\to\infty} \left[\frac{1}{t} \left(t \tan^{-1} t - \tfrac{1}{2} \ln(1+t^2) \right) \right] = \lim_{t\to\infty} \left[\tan^{-1} t - \frac{\ln(1+t^2)}{2t} \right]$$

$$\overset{\text{H}}{=} \frac{\pi}{2} - \lim_{t\to\infty} \frac{2t/(1+t^2)}{2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

(b) $f(x) \geq 0$ and $\int_a^\infty f(x)\,dx$ is divergent $\Rightarrow$ $\lim\limits_{t\to\infty} \int_a^t f(x)\,dx = \infty$.

$$f_{\text{ave}} = \lim_{t\to\infty} \frac{\int_a^t f(x)\,dx}{t-a}\,dx \overset{\text{H}}{=} \lim_{t\to\infty} \frac{f(t)}{1} \quad [\text{by FTC1}] \quad = \lim_{x\to\infty} f(x), \text{ if this limit exists.}$$

(c) Suppose $\int_a^\infty f(x)\,dx$ converges; that is, $\lim\limits_{t\to\infty} \int_a^t f(x)\,dx = L < \infty$. Then

$$f_{\text{ave}} = \lim_{t\to\infty} \left[\frac{1}{t-a} \int_a^t f(x)\,dx \right] = \lim_{t\to\infty} \frac{1}{t-a} \cdot \lim_{t\to\infty} \int_a^t f(x)\,dx = 0 \cdot L = 0.$$

(d) $(\sin x)_{\text{ave}} = \lim\limits_{t\to\infty} \dfrac{1}{t} \displaystyle\int_0^t \sin x\,dx = \lim\limits_{t\to\infty} \left(\dfrac{1}{t} \left[-\cos x \right]_0^t \right) = \lim\limits_{t\to\infty} \left(-\dfrac{\cos t}{t} + \dfrac{1}{t} \right) = \lim\limits_{t\to\infty} \dfrac{1-\cos t}{t} = 0$

79. Let $u = 1/x \;\Rightarrow\; x = 1/u \;\Rightarrow\; dx = -(1/u^2)\,du$.

$$\int_0^\infty \frac{\ln x}{1+x^2}\,dx = \int_\infty^0 \frac{\ln(1/u)}{1+1/u^2} \left(-\frac{du}{u^2} \right) = \int_\infty^0 \frac{-\ln u}{u^2+1}(-du) = \int_\infty^0 \frac{\ln u}{1+u^2}\,du = -\int_0^\infty \frac{\ln u}{1+u^2}\,du$$

Therefore, $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,dx = -\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0.$

80. If the distance between P and the point charge is d, then the potential V at P is

$$V = W = \int_\infty^d F\,dr = \int_\infty^d \frac{q}{4\pi\varepsilon_0 r^2}\,dr = \lim_{t\to\infty} \frac{q}{4\pi\varepsilon_0} \left[-\frac{1}{r} \right]_t^d = \frac{q}{4\pi\varepsilon_0} \lim_{t\to\infty} \left(-\frac{1}{d} + \frac{1}{t} \right) = -\frac{q}{4\pi\varepsilon_0 d}.$$