8 ☐ FURTHER APPLICATIONS OF INTEGRATION

8.1 Arc Length

1. $y = 2x - 5 \quad \Rightarrow \quad L = \int_{-1}^{3} \sqrt{1 + (dy/dx)^2}\, dx = \int_{-1}^{3} \sqrt{1 + (2)^2}\, dx = \sqrt{5}\,[3 - (-1)] = 4\sqrt{5}.$

The arc length can be calculated using the distance formula, since the curve is a line segment, so

$L = [\text{distance from } (-1, -7) \text{ to } (3, 1)] = \sqrt{[3 - (-1)]^2 + [1 - (-7)]^2} = \sqrt{80} = 4\sqrt{5}$

2. Using the arc length formula with $y = \sqrt{2 - x^2} \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{x}{\sqrt{2 - x^2}}$, we get

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + \frac{x^2}{2 - x^2}}\, dx = \int_0^1 \frac{\sqrt{2}\, dx}{\sqrt{2 - x^2}} = \sqrt{2} \int_0^1 \frac{dx}{\sqrt{\left(\sqrt{2}\right)^2 - x^2}}$$

$$= \sqrt{2}\left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^1 = \sqrt{2}\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1} 0\right] = \sqrt{2}\left[\tfrac{\pi}{4} - 0\right] = \sqrt{2}\,\tfrac{\pi}{4}$$

The curve is a one-eighth of a circle with radius $\sqrt{2}$, so the length of the arc is $\frac{1}{8}\left(2\pi \cdot \sqrt{2}\right) = \sqrt{2}\,\frac{\pi}{4}$, as above.

3. $y = \cos x \quad \Rightarrow \quad dy/dx = -\sin x \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + \sin^2 x$. So $L = \int_0^{2\pi} \sqrt{1 + \sin^2 x}\, dx$.

4. $y = xe^{-x^2} \quad \Rightarrow \quad dy/dx = xe^{-x^2}(-2x) + e^{-x^2} \cdot 1 = e^{-x^2}(1 - 2x^2) \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + (1 - 2x^2)^2 e^{-2x^2}$.

So $L = \int_0^1 \sqrt{1 + (1 - 2x^2)^2\, e^{-2x^2}}\, dx$.

5. $x = y + y^3 \quad \Rightarrow \quad dx/dy = 1 + 3y^2 \quad \Rightarrow \quad 1 + (dx/dy)^2 = 1 + (1 + 3y^2)^2 = 9y^4 + 6y^2 + 2$.

So $L = \int_1^4 \sqrt{9y^4 + 6y^2 + 2}\, dy$.

6. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $y = \pm b\sqrt{1 - x^2/a^2} = \pm\dfrac{b}{a}\sqrt{a^2 - x^2}$ [assume $a > 0$]. $y = \dfrac{b}{a}\sqrt{a^2 - x^2} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{-bx}{a\sqrt{a^2 - x^2}} \quad \Rightarrow$

$\left(\dfrac{dy}{dx}\right)^2 = \dfrac{b^2x^2}{a^2(a^2 - x^2)}$. So $L = 2\displaystyle\int_{-a}^{a}\left[1 + \frac{b^2x^2}{a^2(a^2 - x^2)}\right]^{1/2} dx = \frac{4}{a}\int_0^a \left[\frac{(b^2 - a^2)x^2 + a^4}{a^2 - x^2}\right]^{1/2} dx.$

7. $y = 1 + 6x^{3/2} \quad \Rightarrow \quad dy/dx = 9x^{1/2} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + 81x$. So

$L = \int_0^1 \sqrt{1 + 81x}\, dx = \int_1^{82} u^{1/2}\left(\frac{1}{81}\, du\right) \quad \left[\begin{array}{l} u = 1 + 81x, \\ du = 81\, dx \end{array}\right] \quad = \frac{1}{81} \cdot \frac{2}{3}\left[u^{3/2}\right]_1^{82} = \frac{2}{243}\left(82\sqrt{82} - 1\right)$

8. $y^2 = 4(x + 4)^3,\ y > 0 \quad \Rightarrow \quad y = 2(x + 4)^{3/2} \quad \Rightarrow \quad dy/dx = 3(x + 4)^{1/2} \quad \Rightarrow$

$1 + (dy/dx)^2 = 1 + 9(x + 4) = 9x + 37$. So

$L = \int_0^2 \sqrt{9x + 37}\, dx \quad \left[\begin{array}{l} u = 9x + 37, \\ du = 9\, dx \end{array}\right] \quad = \int_{37}^{55} u^{1/2}\left(\frac{1}{9}\, du\right) = \frac{1}{9} \cdot \frac{2}{3}\left[u^{3/2}\right]_{37}^{55} = \frac{2}{27}\left(55\sqrt{55} - 37\sqrt{37}\right).$

9. $y = \dfrac{x^5}{6} + \dfrac{1}{10x^3} \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{5}{6}x^4 - \dfrac{3}{10}x^{-4} \quad\Rightarrow$

$1 + (dy/dx)^2 = 1 + \frac{25}{36}x^8 - \frac{1}{2} + \frac{9}{100}x^{-8} = \frac{25}{36}x^8 + \frac{1}{2} + \frac{9}{100}x^{-8} = \left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right)^2$. So

$L = \int_1^2 \sqrt{\left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right)^2}\,dx = \int_1^2 \left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right)dx = \left[\frac{1}{6}x^5 - \frac{1}{10}x^{-3}\right]_1^2 = \left(\frac{32}{6} - \frac{1}{80}\right) - \left(\frac{1}{6} - \frac{1}{10}\right)$
$= \frac{31}{6} + \frac{7}{80} = \frac{1261}{240}$

10. $x = \dfrac{y^4}{8} + \dfrac{1}{4y^2} \quad\Rightarrow\quad \dfrac{dx}{dy} = \frac{1}{2}y^3 - \frac{1}{2}y^{-3} \quad\Rightarrow$

$1 + (dx/dy)^2 = 1 + \frac{1}{4}y^6 - \frac{1}{2} + \frac{1}{4}y^{-6} = \frac{1}{4}y^6 + \frac{1}{2} + \frac{1}{4}y^{-6} = \left(\frac{1}{2}y^3 + \frac{1}{2}y^{-3}\right)^2$. So

$L = \int_1^2 \sqrt{\left(\frac{1}{2}y^3 + \frac{1}{2}y^{-3}\right)^2}\,dy = \int_1^2 \left(\frac{1}{2}y^3 + \frac{1}{2}y^{-3}\right)dy = \left[\frac{1}{8}y^4 - \frac{1}{4}y^{-2}\right]_1^2 = \left(2 - \frac{1}{16}\right) - \left(\frac{1}{8} - \frac{1}{4}\right)$
$= 2 + \frac{1}{16} = \frac{33}{16}$.

11. $x = \frac{1}{3}\sqrt{y}\,(y-3) = \frac{1}{3}y^{3/2} - y^{1/2} \quad\Rightarrow\quad dx/dy = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \quad\Rightarrow$

$1 + (dx/dy)^2 = 1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1} = \frac{1}{4}y + \frac{1}{2} + \frac{1}{4}y^{-1} = \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2$. So

$L = \int_1^9 \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)dy = \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9 = \frac{1}{2}\left[\left(\frac{2}{3}\cdot 27 + 2\cdot 3\right) - \left(\frac{2}{3}\cdot 1 + 2\cdot 1\right)\right]$
$= \frac{1}{2}\left(24 - \frac{8}{3}\right) = \frac{1}{2}\left(\frac{64}{3}\right) = \frac{32}{3}$.

12. $y = \ln(\cos x) \quad\Rightarrow\quad dy/dx = -\tan x \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + \tan^2 x = \sec^2 x$. So

$L = \int_0^{\pi/3} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/3} \sec x\,dx = \left[\ln|\sec x + \tan x|\right]_0^{\pi/3} = \ln\left(2 + \sqrt{3}\right) - \ln(1+0) = \ln\left(2+\sqrt{3}\right)$.

13. $y = \ln(\sec x) \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{\sec x\,\tan x}{\sec x} = \tan x \quad\Rightarrow\quad 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$, so

$L = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/4} |\sec x|\,dx = \int_0^{\pi/4} \sec x\,dx = \Big[\ln(\sec x + \tan x)\Big]_0^{\pi/4}$
$= \ln\left(\sqrt{2}+1\right) - \ln(1+0) = \ln\left(\sqrt{2}+1\right)$

14. $y = 3 + \frac{1}{2}\cosh 2x \quad\Rightarrow\quad y' = \sinh 2x \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + \sinh^2(2x) = \cosh^2(2x)$. So

$L = \int_0^1 \sqrt{\cosh^2(2x)}\,dx = \int_0^1 \cosh 2x\,dx = \left[\frac{1}{2}\sinh 2x\right]_0^1 = \frac{1}{2}\sinh 2 - 0 = \frac{1}{2}\sinh 2$.

15. $y = \ln(1 - x^2) \quad\Rightarrow\quad y' = \dfrac{1}{1-x^2}\cdot(-2x) \quad\Rightarrow$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4x^2}{(1-x^2)^2} = \frac{1 - 2x^2 + x^4 + 4x^2}{(1-x^2)^2} = \frac{1 + 2x^2 + x^4}{(1-x^2)^2} = \frac{(1+x^2)^2}{(1-x^2)^2} \quad\Rightarrow$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{\left(\frac{1+x^2}{1-x^2}\right)^2} = \frac{1+x^2}{1-x^2} = -1+\frac{2}{1-x^2} \quad \text{[by division]} \quad = -1+\frac{1}{1+x}+\frac{1}{1-x} \quad \text{[partial fractions]}.$$

So $L = \int_0^{1/2}\left(-1+\frac{1}{1+x}+\frac{1}{1-x}\right)dx = \left[-x+\ln|1+x|-\ln|1-x|\right]_0^{1/2} = \left(-\frac{1}{2}+\ln\frac{3}{2}-\ln\frac{1}{2}\right)-0 = \ln 3-\frac{1}{2}.$

16. $y=\sqrt{x-x^2}+\sin^{-1}\left(\sqrt{x}\right) \Rightarrow \frac{dy}{dx} = \frac{1-2x}{2\sqrt{x-x^2}}+\frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{2-2x}{2\sqrt{x}\sqrt{1-x}} = \sqrt{\frac{1-x}{x}} \Rightarrow$

$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{1-x}{x} = \frac{1}{x}$. The curve has endpoints $(0,0)$ and $\left(1,\frac{\pi}{2}\right)$, so $L = \int_0^1 \sqrt{\frac{1}{x}}\,dx = \left[2\sqrt{x}\right]_0^1 = 2.$

17. $y=e^x \Rightarrow y'=e^x \Rightarrow 1+(y')^2 = 1+e^{2x}$. So

$$\begin{aligned} L &= \int_0^1 \sqrt{1+e^{2x}}\,dx = \int_1^e \sqrt{1+u^2}\,\frac{du}{u} \quad \left[\begin{array}{c} u=e^x,\text{ so} \\ x=\ln u,\ dx=du/u \end{array}\right] \quad = \int_1^e \frac{\sqrt{1+u^2}}{u^2}\,u\,du \\ &= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{v}{v^2-1}\,v\,dv \quad \left[\begin{array}{c} v=\sqrt{1+u^2},\text{ so} \\ v^2=1+u^2,\ v\,dv=u\,du \end{array}\right] = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1+\frac{1/2}{v-1}-\frac{1/2}{v+1}\right)dv \\ &= \left[v+\frac{1}{2}\ln\frac{v-1}{v+1}\right]_{\sqrt{2}}^{\sqrt{1+e^2}} = \sqrt{1+e^2}+\frac{1}{2}\ln\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}-\sqrt{2}-\frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1} \\ &= \sqrt{1+e^2}-\sqrt{2}+\ln\left(\sqrt{1+e^2}-1\right)-1-\ln\left(\sqrt{2}-1\right) \end{aligned}$$

Or: Use Formula 23 for $\int\left(\sqrt{1+u^2}/u\right)du$, or substitute $u=\tan\theta$.

18. $y=\ln\left(\frac{e^x+1}{e^x-1}\right) = \ln(e^x+1)-\ln(e^x-1) \Rightarrow y' = \frac{e^x}{e^x+1}-\frac{e^x}{e^x-1} = \frac{-2e^x}{e^{2x}-1} \Rightarrow$

$1+(y')^2 = 1+\frac{4e^{2x}}{(e^{2x}-1)^2} = \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2} \Rightarrow \sqrt{1+(y')^2} = \frac{e^{2x}+1}{e^{2x}-1} = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{\cosh x}{\sinh x}.$

So $L = \int_a^b \frac{\cosh x}{\sinh x}\,dx = \left[\ln\sinh x\right]_a^b = \ln\sinh b-\ln\sinh a = \ln\left(\frac{\sinh b}{\sinh a}\right) = \ln\left(\frac{e^b-e^{-b}}{e^a-e^{-a}}\right).$

19. $y=\frac{1}{2}x^2 \Rightarrow dy/dx = x \Rightarrow 1+(dy/dx)^2 = 1+x^2$. So

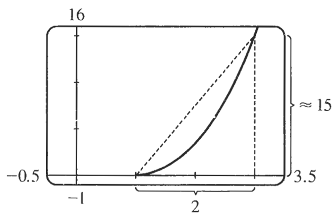
$$\begin{aligned} L &= \int_{-1}^1 \sqrt{1+x^2}\,dx = 2\int_0^1 \sqrt{1+x^2}\,dx \quad \text{[by symmetry]} \quad \overset{21}{=} 2\left[\tfrac{x}{2}\sqrt{1+x^2}+\tfrac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)\right]_0^1 \quad \left[\begin{array}{c}\text{or substitute} \\ x=\tan\theta\end{array}\right] \\ &= 2\left[\left(\tfrac{1}{2}\sqrt{2}+\tfrac{1}{2}\ln\left(1+\sqrt{2}\right)\right)-\left(0+\tfrac{1}{2}\ln 1\right)\right] = \sqrt{2}+\ln\left(1+\sqrt{2}\right) \end{aligned}$$

20. $x^2=(y-4)^3 \Rightarrow x=(y-4)^{3/2}$ [for $x>0$] $\Rightarrow dx/dy = \frac{3}{2}(y-4)^{1/2} \Rightarrow$

$1+(dx/dy)^2 = 1+\frac{9}{4}(y-4) = \frac{9}{4}y-8$. So

$$\begin{aligned} L &= \int_5^8 \sqrt{\tfrac{9}{4}y-8}\,dy = \int_{13/4}^{10} \sqrt{u}\left(\tfrac{4}{9}\,du\right) \quad \left[\begin{array}{c} u=\frac{9}{4}y-8, \\ du=\frac{9}{4}\,dy \end{array}\right] \quad = \tfrac{4}{9}\left[\tfrac{2}{3}u^{3/2}\right]_{13/4}^{10} \\ &= \tfrac{8}{27}\left[10^{3/2}-\left(\tfrac{13}{4}\right)^{3/2}\right] \quad \left[\text{or } \tfrac{1}{27}\left(80\sqrt{10}-13\sqrt{13}\right)\right] \end{aligned}$$

21.



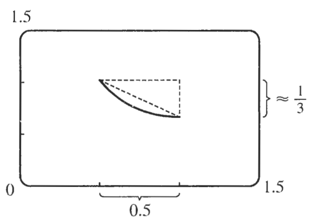
From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(1, 0)$, $(3, 0)$, and $(3, f(3)) \approx (3, 15)$, where $y = f(x) = \frac{2}{3}(x^2 - 1)^{3/2}$. This length is about $\sqrt{15^2 + 2^2} \approx 15$, so we might estimate the length to be 15.5.

$y = \frac{2}{3}(x^2 - 1)^{3/2} \quad \Rightarrow \quad y' = (x^2 - 1)^{1/2}(2x) \quad \Rightarrow \quad 1 + (y')^2 = 1 + 4x^2(x^2 - 1) = 4x^4 - 4x^2 + 1 = (2x^2 - 1)^2$,

so, using the fact that $2x^2 - 1 > 0$ for $1 \le x \le 3$,

$L = \int_1^3 \sqrt{(2x^2 - 1)^2}\,dx = \int_1^3 \left|2x^2 - 1\right|dx = \int_1^3 (2x^2 - 1)\,dx = \left[\frac{2}{3}x^3 - x\right]_1^3 = (18 - 3) - \left(\frac{2}{3} - 1\right) = \frac{46}{3} = 15.\overline{3}$.

22.



From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(0.5, f(0.5) \approx 1)$, $(1, f(0.5) \approx 1)$ and $\left(1, \frac{2}{3}\right)$, where $y = f(x) = x^3/6 + 1/(2x)$. This length is about $\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2} \approx 0.6$, so we might estimate the length to be 0.65.

$$y = \frac{x^3}{6} + \frac{1}{2x} \quad \Rightarrow \quad y' = \frac{x^2}{2} - \frac{x^{-2}}{2} \quad \Rightarrow \quad 1 + (y')^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{x^{-4}}{4} = \frac{x^4}{4} + \frac{1}{2} + \frac{x^{-4}}{4} = \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2$$

so, using the fact that the parenthetical expression is positive,

$$L = \int_{1/2}^1 \sqrt{\left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2}\,dx = \int_{1/2}^1 \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_{1/2}^1 = \left(\frac{1}{6} - \frac{1}{2}\right) - \left(\frac{1}{48} - 1\right) = \frac{31}{48} = 0.6458\overline{3}$$

23. $y = xe^{-x} \quad \Rightarrow \quad dy/dx = e^{-x} - xe^{-x} = e^{-x}(1 - x) \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + e^{-2x}(1 - x)^2$. Let $f(x) = \sqrt{1 + (dy/dx)^2} = \sqrt{1 + e^{-2x}(1 - x)^2}$. Then $L = \int_0^5 f(x)\,dx$. Since $n = 10$, $\Delta x = \frac{5 - 0}{10} = \frac{1}{2}$. Now

$$L \approx S_{10} = \frac{1/2}{3}\left[f(0) + 4f\left(\tfrac{1}{2}\right) + 2f(1) + 4f\left(\tfrac{3}{2}\right) + 2f(2) + 4f\left(\tfrac{5}{2}\right) + 2f(3) + 4f\left(\tfrac{7}{2}\right) + 2f(4) + 4f\left(\tfrac{9}{2}\right) + f(5)\right] \approx 5.115840$$

The value of the integral produced by a calculator is 5.113568 (to six decimal places).

24. $x = y + \sqrt{y} \quad \Rightarrow \quad dx/dy = 1 + \dfrac{1}{2\sqrt{y}} \quad \Rightarrow \quad 1 + (dx/dy)^2 = 1 + \left(1 + \dfrac{1}{2\sqrt{y}}\right)^2 = 2 + \dfrac{1}{\sqrt{y}} + \dfrac{1}{4y}$.

Let $g(y) = \sqrt{1 + (dx/dy)^2}$. Then $L = \int_1^2 g(y)\,dy$. Since $n = 10$, $\Delta y = \frac{2 - 1}{10} = \frac{1}{10}$. Now

$$L \approx S_{10} = \frac{1/10}{3}[g(1) + 4g(1.1) + 2g(1.2) + 4g(1.3) + 2g(1.4) + 4g(1.5) + 2g(1.6) + 4g(1.7) + 2g(1.8) + 4g(1.9) + g(2)] \approx 1.732215,$$

which is the same value of the integral produced by a calculator to six decimal places.

25. $y = \sec x \;\Rightarrow\; dy/dx = \sec x\,\tan x \;\Rightarrow\; L = \int_0^{\pi/3} f(x)\,dx$, where $f(x) = \sqrt{1 + \sec^2 x\,\tan^2 x}$.

Since $n = 10$, $\Delta x = \dfrac{\pi/3 - 0}{10} = \dfrac{\pi}{30}$. Now

$$L \approx S_{10} = \frac{\pi/30}{3}\left[f(0) + 4f\left(\frac{\pi}{30}\right) + 2f\left(\frac{2\pi}{30}\right) + 4f\left(\frac{3\pi}{30}\right) + 2f\left(\frac{4\pi}{30}\right) + 4f\left(\frac{5\pi}{30}\right)\right.$$
$$\left. + 2f\left(\frac{6\pi}{30}\right) + 4f\left(\frac{7\pi}{30}\right) + 2f\left(\frac{8\pi}{30}\right) + 4f\left(\frac{9\pi}{30}\right) + f\left(\frac{\pi}{3}\right)\right] \approx 1.569619.$$

The value of the integral produced by a calculator is 1.569259 (to six decimal places).

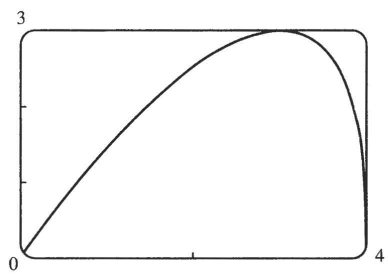
26. $y = x\ln x \;\Rightarrow\; dy/dx = 1 + \ln x$. Let $f(x) = \sqrt{1 + (dy/dx)^2} = \sqrt{1 + (1 + \ln x)^2}$. Then $L = \int_1^3 f(x)\,dx$.

Since $n = 10$, $\Delta x = \frac{3-1}{10} = \frac{1}{5}$. Now

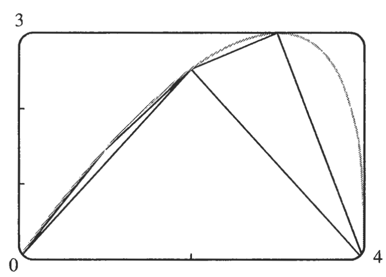
$$L \approx S_{10} = \tfrac{1/5}{3}[f(1) + 4f(1.2) + 2f(1.4) + 4f(1.6) + 2f(1.8) + 4f(2)$$
$$+ 2f(2.2) + 4f(2.4) + 2f(2.6) + 4f(2.8) + f(3)] \approx 3.869618.$$

The value of the integral produced by a calculator is 3.869617 (to six decimal places).

27. (a)



(b)



Let $f(x) = y = x\sqrt[3]{4-x}$. The polygon with one side is just the line segment joining the points $(0, f(0)) = (0,0)$ and $(4, f(4)) = (4,0)$, and its length $L_1 = 4$.

The polygon with two sides joins the points $(0,0)$, $(2, f(2)) = \left(2, 2\sqrt[3]{2}\right)$ and $(4,0)$. Its length

$$L_2 = \sqrt{(2-0)^2 + \left(2\sqrt[3]{2} - 0\right)^2} + \sqrt{(4-2)^2 + \left(0 - 2\sqrt[3]{2}\right)^2} = 2\sqrt{4 + 2^{8/3}} \approx 6.43$$

Similarly, the inscribed polygon with four sides joins the points $(0,0)$, $\left(1, \sqrt[3]{3}\right)$, $\left(2, 2\sqrt[3]{2}\right)$, $(3,3)$, and $(4,0)$, so its length

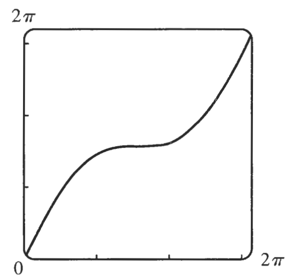
$$L_3 = \sqrt{1 + \left(\sqrt[3]{3}\right)^2} + \sqrt{1 + \left(2\sqrt[3]{2} - \sqrt[3]{3}\right)^2} + \sqrt{1 + \left(3 - 2\sqrt[3]{2}\right)^2} + \sqrt{1+9} \approx 7.50$$

(c) Using the arc length formula with $\dfrac{dy}{dx} = x\left[\frac{1}{3}(4-x)^{-2/3}(-1)\right] + \sqrt[3]{4-x} = \dfrac{12 - 4x}{3(4-x)^{2/3}}$, the length of the curve is

$$L = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^4 \sqrt{1 + \left[\frac{12 - 4x}{3(4-x)^{2/3}}\right]^2}\,dx.$$

(d) According to a CAS, the length of the curve is $L \approx 7.7988$. The actual value is larger than any of the approximations in part (b). This is always true, since any approximating straight line between two points on the curve is shorter than the length of the curve between the two points.

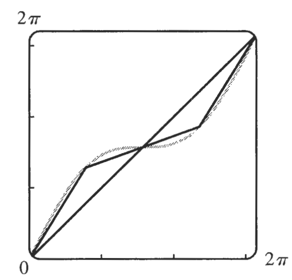
28. (a) Let $f(x) = y = x + \sin x$ with $0 \le x \le 2\pi$.



(b) The polygon with one side is just the line segment joining the points $(0, f(0)) = (0, 0)$ and $(2\pi, f(2\pi)) = (2\pi, 2\pi)$, and its length is $\sqrt{(2\pi - 0)^2 + (2\pi - 0)^2} = 2\sqrt{2}\,\pi \approx 8.9$.

The polygon with two sides joins the points $(0, 0)$, $(\pi, f(\pi)) = (\pi, \pi)$, and $(2\pi, 2\pi)$. Its length is

$$\sqrt{(\pi - 0)^2 + (\pi - 0)^2} + \sqrt{(2\pi - \pi)^2 + (2\pi - \pi)^2} = \sqrt{2}\,\pi + \sqrt{2}\,\pi$$
$$= 2\sqrt{2}\,\pi \approx 8.9$$



Note from the diagram that the two approximations are the same because the sides of the two-sided polygon are in fact on the same line, since $f(\pi) = \pi = \frac{1}{2}f(2\pi)$.

The four-sided polygon joins the points $(0, 0)$, $\left(\frac{\pi}{2}, \frac{\pi}{2} + 1\right)$, (π, π), $\left(\frac{3\pi}{2}, \frac{3\pi}{2} - 1\right)$, and $(2\pi, 2\pi)$, so its length is

$$\sqrt{\left(\tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2} + 1\right)^2} + \sqrt{\left(\tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2} - 1\right)^2} + \sqrt{\left(\tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2} - 1\right)^2} + \sqrt{\left(\tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2} + 1\right)^2} \approx 9.4$$

(c) Using the arc length formula with $dy/dx = 1 + \cos x$, the length of the curve is

$$L = \int_0^{2\pi} \sqrt{1 + (1 + \cos x)^2}\, dx = \int_0^{2\pi} \sqrt{2 + 2\cos x + \cos^2 x}\, dx$$

(d) The CAS approximates the integral as 9.5076. The actual length is larger than the approximations in part (b).

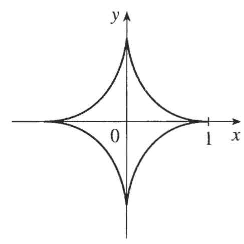
29. $y = \ln x \;\Rightarrow\; dy/dx = 1/x \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + 1/x^2 = (x^2 + 1)/x^2 \;\Rightarrow$

$$L = \int_1^2 \sqrt{\frac{x^2 + 1}{x^2}}\, dx = \int_1^2 \frac{\sqrt{1 + x^2}}{x}\, dx \overset{23}{=} \left[\sqrt{1 + x^2} - \ln\left|\frac{1 + \sqrt{1 + x^2}}{x}\right|\right]_1^2$$
$$= \sqrt{5} - \ln\left(\frac{1 + \sqrt{5}}{2}\right) - \sqrt{2} + \ln\left(1 + \sqrt{2}\right)$$

30. $y = x^{4/3} \;\Rightarrow\; dy/dx = \frac{4}{3}x^{1/3} \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + \frac{16}{9}x^{2/3} \;\Rightarrow$

$$L = \int_0^1 \sqrt{1 + \tfrac{16}{9}x^{2/3}}\, dx = \int_0^{4/3} \sqrt{1 + u^2}\, \tfrac{81}{64}u^2\, du \qquad \left[\begin{array}{l} u = \frac{4}{3}x^{1/3},\ du = \frac{4}{9}x^{-2/3}\,dx, \\ dx = \frac{9}{4}x^{2/3}\,du = \frac{9}{4}\cdot\frac{9}{16}u^2\,du = \frac{81}{64}u^2\,du \end{array}\right]$$

$$\overset{22}{=} \tfrac{81}{64}\left[\tfrac{1}{8}u(1 + 2u^2)\sqrt{1 + u^2} - \tfrac{1}{8}\ln\left(u + \sqrt{1 + u^2}\right)\right]_0^{4/3} = \tfrac{81}{64}\left[\tfrac{1}{6}\left(1 + \tfrac{32}{9}\right)\sqrt{\tfrac{25}{9}} - \tfrac{1}{8}\ln\left(\tfrac{4}{3} + \sqrt{\tfrac{25}{9}}\right)\right]$$
$$= \tfrac{81}{64}\left(\tfrac{1}{6}\cdot\tfrac{41}{9}\cdot\tfrac{5}{3} - \tfrac{1}{8}\ln 3\right) = \tfrac{205}{128} - \tfrac{81}{512}\ln 3 \approx 1.4277586$$



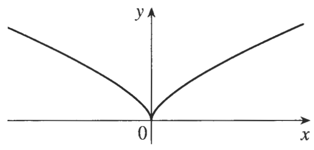
31. $y^{2/3} = 1 - x^{2/3} \Rightarrow y = (1 - x^{2/3})^{3/2} \Rightarrow$

$\dfrac{dy}{dx} = \frac{3}{2}(1 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}(1 - x^{2/3})^{1/2} \Rightarrow$

$\left(\dfrac{dy}{dx}\right)^2 = x^{-2/3}(1 - x^{2/3}) = x^{-2/3} - 1$. Thus

$L = 4\int_0^1 \sqrt{1 + (x^{-2/3} - 1)}\,dx = 4\int_0^1 x^{-1/3}\,dx = 4 \lim\limits_{t \to 0^+} \left[\frac{3}{2}x^{2/3}\right]_t^1 = 6.$

32. (a)



(b) $y = x^{2/3} \Rightarrow 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \left(\frac{2}{3}x^{-1/3}\right)^2 = 1 + \frac{4}{9}x^{-2/3}$. So $L = \int_0^1 \sqrt{1 + \frac{4}{9}x^{-2/3}}\,dx$ [an improper integral].

$x = y^{3/2} \Rightarrow 1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + \left(\frac{3}{2}y^{1/2}\right)^2 = 1 + \frac{9}{4}y$. So $L = \int_0^1 \sqrt{1 + \frac{9}{4}y}\,dy$.

The second integral equals $\frac{4}{9} \cdot \frac{2}{3}\left[\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^1 = \frac{8}{27}\left(\frac{13\sqrt{13}}{8} - 1\right) = \frac{13\sqrt{13} - 8}{27}$.

The first integral can be evaluated as follows:

$$\int_0^1 \sqrt{1 + \tfrac{4}{9}x^{-2/3}}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}\,dx = \lim_{t \to 0^+} \int_{9t^{2/3}}^9 \frac{\sqrt{u + 4}}{18}\,du \qquad \left[\begin{array}{l} u = 9x^{2/3}, \\ du = 6x^{-1/3}\,dx \end{array}\right]$$

$$= \int_0^9 \frac{\sqrt{u + 4}}{18}\,du = \frac{1}{18} \cdot \left[\frac{2}{3}(u + 4)^{3/2}\right]_0^9 = \frac{1}{27}(13^{3/2} - 4^{3/2}) = \frac{13\sqrt{13} - 8}{27}$$

(c) $L =$ length of the arc of this curve from $(-1, 1)$ to $(8, 4)$

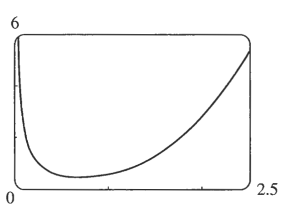
$$= \int_0^1 \sqrt{1 + \frac{9}{4}y}\,dy + \int_0^4 \sqrt{1 + \frac{9}{4}y}\,dy = \frac{13\sqrt{13} - 8}{27} + \frac{8}{27}\left[\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^4 \qquad \text{[from part (b)]}$$

$$= \frac{13\sqrt{13} - 8}{27} + \frac{8}{27}(10\sqrt{10} - 1) = \frac{13\sqrt{13} + 80\sqrt{10} - 16}{27}$$

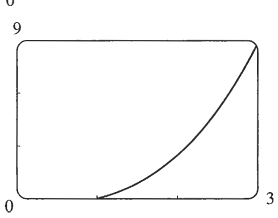
33. $y = 2x^{3/2} \Rightarrow y' = 3x^{1/2} \Rightarrow 1 + (y')^2 = 1 + 9x$. The arc length function with starting point $P_0(1, 2)$ is

$s(x) = \int_1^x \sqrt{1 + 9t}\,dt = \left[\frac{2}{27}(1 + 9t)^{3/2}\right]_1^x = \frac{2}{27}\left[(1 + 9x)^{3/2} - 10\sqrt{10}\right]$.

34. (a)



(c)



(b) $f(x) = \dfrac{1}{3}x^3 + \dfrac{1}{4x}$, $x > 0 \Rightarrow f'(x) = x^2 - \dfrac{1}{4x^2}$. Then

$$1 + [f'(x)]^2 = 1 + \left(x^4 - \frac{1}{2} + \frac{1}{16x^4}\right) = x^4 + \frac{1}{2} + \frac{1}{16x^4} = \left(x^2 + \frac{1}{4x^2}\right)^2,$$

so $\sqrt{1 + [f'(x)]^2} = x^2 + \dfrac{1}{4x^2}$. Thus,

$$s(x) = \int_1^x \sqrt{1 + [f'(t)]^2}\,dt = \int_1^x \left(t^2 + \frac{1}{4t^2}\right)dt = \left[\frac{1}{3}t^3 - \frac{1}{4t}\right]_1^x$$

$$= \left(\frac{1}{3}x^3 - \frac{1}{4x}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{3}x^3 - \frac{1}{4x} - \frac{1}{12} \quad \text{for } x \ge 1$$

35. $y = \sin^{-1} x + \sqrt{1 - x^2} \quad \Rightarrow \quad y' = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{x}{\sqrt{1-x^2}} = \dfrac{1-x}{\sqrt{1-x^2}} \quad \Rightarrow$

$$1 + (y')^2 = 1 + \frac{(1-x)^2}{1-x^2} = \frac{1 - x^2 + 1 - 2x + x^2}{1-x^2} = \frac{2-2x}{1-x^2} = \frac{2(1-x)}{(1+x)(1-x)} = \frac{2}{1+x} \quad \Rightarrow$$

$\sqrt{1+(y')^2} = \sqrt{\dfrac{2}{1+x}}$. Thus, the arc length function with starting point $(0,1)$ is given by

$$s(x) = \int_0^x \sqrt{1 + [f'(t)]^2}\, dt = \int_0^x \sqrt{\frac{2}{1+t}}\, dt = \sqrt{2}\left[2\sqrt{1+t}\,\right]_0^x = 2\sqrt{2}\left(\sqrt{1+x} - 1\right).$$

36. $y = 150 - \frac{1}{40}(x-50)^2 \quad \Rightarrow \quad y' = -\frac{1}{20}(x-50) \quad \Rightarrow \quad 1 + (y')^2 = 1 + \frac{1}{20^2}(x-50)^2$, so the distance traveled by the kite is

$$L = \int_0^{80} \sqrt{1 + \frac{1}{20^2}(x-50)^2}\, dx = \int_{-5/2}^{3/2} \sqrt{1+u^2}\,(20\, du) \qquad \begin{bmatrix} u = \frac{1}{20}(x-50), \\ du = \frac{1}{20}\, dx \end{bmatrix}$$

$$\overset{21}{=} 20\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_{-5/2}^{3/2} = 10\left[\tfrac{3}{2}\sqrt{\tfrac{13}{4}} + \ln\left(\tfrac{3}{2} + \sqrt{\tfrac{13}{4}}\right) + \tfrac{5}{2}\sqrt{\tfrac{29}{4}} - \ln\left(-\tfrac{5}{2} + \sqrt{\tfrac{29}{4}}\right)\right]$$

$$= \tfrac{15}{2}\sqrt{13} + \tfrac{25}{2}\sqrt{29} + 10\ln\left(\frac{3+\sqrt{13}}{-5+\sqrt{29}}\right) \approx 122.8 \text{ ft}$$

37. The prey hits the ground when $y = 0 \quad \Leftrightarrow \quad 180 - \frac{1}{45}x^2 = 0 \quad \Leftrightarrow \quad x^2 = 45 \cdot 180 \quad \Rightarrow \quad x = \sqrt{8100} = 90$, since x must be positive. $y' = -\frac{2}{45}x \quad \Rightarrow \quad 1 + (y')^2 = 1 + \frac{4}{45^2}x^2$, so the distance traveled by the prey is

$$L = \int_0^{90} \sqrt{1 + \frac{4}{45^2}x^2}\, dx = \int_0^4 \sqrt{1+u^2}\left(\tfrac{45}{2}\, du\right) \qquad \begin{bmatrix} u = \frac{2}{45}x, \\ du = \frac{2}{45}\, dx \end{bmatrix}$$

$$\overset{21}{=} \tfrac{45}{2}\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_0^4 = \tfrac{45}{2}\left[2\sqrt{17} + \tfrac{1}{2}\ln\left(4 + \sqrt{17}\right)\right] = 45\sqrt{17} + \tfrac{45}{4}\ln\left(4+\sqrt{17}\right) \approx 209.1 \text{ m}$$

38. Let $y = a - b\cosh cx$, where $a = 211.49$, $b = 20.96$, and $c = 0.03291765$. Then $y' = -bc\sinh cx \quad \Rightarrow$ $1 + (y')^2 = 1 + b^2c^2\sinh^2(cx)$. So $L = \int_{-91.2}^{91.2} \sqrt{1 + b^2c^2\sinh^2(cx)}\, dx \approx 451.137 \approx 451$, to the nearest meter.

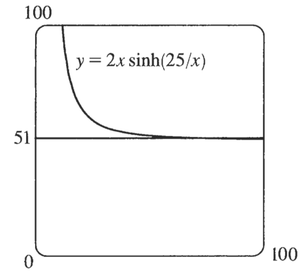
39. The sine wave has amplitude 1 and period 14, since it goes through two periods in a distance of 28 in., so its equation is $y = 1\sin\left(\frac{2\pi}{14}x\right) = \sin\left(\frac{\pi}{7}x\right)$. The width w of the flat metal sheet needed to make the panel is the arc length of the sine curve from $x = 0$ to $x = 28$. We set up the integral to evaluate w using the arc length formula with $\frac{dy}{dx} = \frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)$:

$L = \int_0^{28} \sqrt{1 + \left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\, dx = 2\int_0^{14} \sqrt{1 + \left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\, dx$. This integral would be very difficult to evaluate exactly, so we use a CAS, and find that $L \approx 29.36$ inches.

40. (a) $y = c + a\cosh\left(\frac{x}{a}\right) \quad \Rightarrow \quad y' = \sinh\left(\frac{x}{a}\right) \quad \Rightarrow \quad 1 + (y')^2 = 1 + \sinh^2\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right)$. So

$$L = \int_{-b}^{b} \sqrt{\cosh^2\left(\tfrac{x}{a}\right)}\, dx = 2\int_0^b \cosh\left(\tfrac{x}{a}\right) dx = 2\left[a\sinh\left(\tfrac{x}{a}\right)\right]_0^b = 2a\sinh\left(\tfrac{b}{a}\right).$$

(b) At $x = 0$, $y = c + a$, so $c + a = 20$. The poles are 50 ft apart, so $b = 25$, and $L = 51 \;\Rightarrow\; 51 = 2a\sinh(b/a)$ [from part (a)]. From the figure, we see that $y = 51$ intersects $y = 2x\sinh(25/x)$ at $x \approx 72.3843$ for $x > 0$. So $a \approx 72.3843$ and the wire should be attached at a distance of $y = c + a\cosh(25/a) = 20 - a + a\cosh(25/a) \approx 24.36$ ft above the ground.



41. $y = \int_1^x \sqrt{t^3 - 1}\,dt \;\Rightarrow\; dy/dx = \sqrt{x^3 - 1}$ [by FTC1] $\;\Rightarrow\; 1 + (dy/dx)^2 = 1 + \left(\sqrt{x^3 - 1}\right)^2 = x^3 \;\Rightarrow$

$L = \int_1^4 \sqrt{x^3}\,dx = \int_1^4 x^{3/2}\,dx = \frac{2}{5}\left[x^{5/2}\right]_1^4 = \frac{2}{5}(32 - 1) = \frac{62}{5} = 12.4$

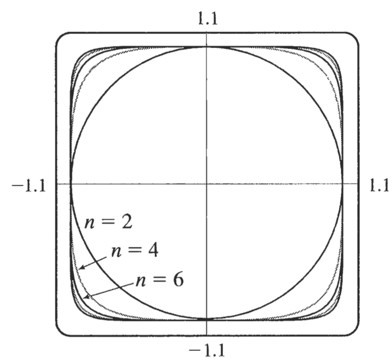
42. By symmetry, the length of the curve in each quadrant is the same, so we'll find the length in the first quadrant and multiply by 4.

$x^{2k} + y^{2k} = 1 \;\Rightarrow\; y^{2k} = 1 - x^{2k} \;\Rightarrow\; y = (1 - x^{2k})^{1/(2k)}$

(in the first quadrant), so we use the arc length formula with

$\dfrac{dy}{dx} = \dfrac{1}{2k}(1 - x^{2k})^{1/(2k)-1}(-2kx^{2k-1}) = -x^{2k-1}(1 - x^{2k})^{1/(2k)-1}$

The total length is therefore



$$L_{2k} = 4\int_0^1 \sqrt{1 + [-x^{2k-1}(1 - x^{2k})^{1/(2k)-1}]^2}\,dx = 4\int_0^1 \sqrt{1 + x^{2(2k-1)}(1 - x^{2k})^{1/k-2}}\,dx$$

Now from the graph, we see that as k increases, the "corners" of these fat circles get closer to the points $(\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$, and the "edges" of the fat circles approach the lines joining these four points. It seems plausible that as $k \to \infty$, the total length of the fat circle with $n = 2k$ will approach the length of the perimeter of the square with sides of length 2. This is supported by taking the limit as $k \to \infty$ of the equation of the fat circle in the first quadrant: $\lim\limits_{k\to\infty} (1 - x^{2k})^{1/(2k)} = 1$ for $0 \le x < 1$. So we guess that $\lim\limits_{k\to\infty} L_{2k} = 4 \cdot 2 = 8$.

DISCOVERY PROJECT Arc Length Contest

For advice on how to run the contest and a list of student entries, see the article "Arc Length Contest" by Larry Riddle in *The College Mathematics Journal*, Volume 29, No. 4, September 1998, pages 314–320.