8.2 Area of a Surface of Revolution

1. $y = x^4 \Rightarrow dy/dx = 4x^3 \Rightarrow ds = \sqrt{1 + (dy/dx)^2}\, dx = \sqrt{1 + 16x^6}\, dx$

(a) By (7), an integral for the area of the surface obtained by rotating the curve about the x-axis is $S = \int 2\pi y\, ds = \int_0^1 2\pi x^4 \sqrt{1 + 16x^6}\, dx$.

(b) By (8), an integral for the area of the surface obtained by rotating the curve about the y-axis is

$S = \int 2\pi x\, ds = \int_0^1 2\pi x \sqrt{1 + 16x^6}\, dx$.

2. $y = xe^{-x} \Rightarrow dy/dx = x(-e^{-x}) + e^{-x} = e^{-x}(1 - x) \Rightarrow ds = \sqrt{1 + (dy/dx)^2}\, dx = \sqrt{1 + e^{-2x}(1 - x)^2}\, dx$

(a) By (7), $S = \int 2\pi y\, ds = \int_1^3 2\pi x e^{-x} \sqrt{1 + e^{-2x}(1 - x)^2}\, dx$.

(b) By (8), $S = \int 2\pi x\, ds = \int_1^3 2\pi x \sqrt{1 + e^{-2x}(1 - x)^2}\, dx$.

3. $y = \tan^{-1} x \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1 + x^2} \Rightarrow ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \dfrac{1}{(1 + x^2)^2}}\, dx.$

(a) By (7), $S = \int 2\pi y\, ds = \int_0^1 2\pi \tan^{-1} x \sqrt{1 + \dfrac{1}{(1 + x^2)^2}}\, dx$.

(b) By (8), $S = \int 2\pi x\, ds = \int_0^1 2\pi x \sqrt{1 + \dfrac{1}{(1 + x^2)^2}}\, dx$.

4. $x = \sqrt{y - y^2}$ [defined for $0 \le y \le 1$] $\Rightarrow$

$$\frac{dx}{dy} = \frac{1 - 2y}{2\sqrt{y - y^2}} \Rightarrow ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{\frac{4(y - y^2) + 1 - 4y + 4y^2}{4(y - y^2)}}\, dy = \sqrt{\frac{1}{4y(1 - y)}}\, dy.$$

(a) By (7), $S = \int 2\pi y\, ds = \int_0^1 2\pi y \sqrt{\dfrac{1}{4y(1 - y)}}\, dy$.

(b) By (8), $S = \int 2\pi x\, ds = \int_0^1 2\pi \sqrt{y - y^2} \sqrt{\dfrac{1}{4y(1 - y)}}\, dy$.

5. $y = x^3 \Rightarrow y' = 3x^2$. So

$$\begin{aligned} S &= \int_0^2 2\pi y \sqrt{1 + (y')^2}\, dx = 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4}\, dx \qquad [u = 1 + 9x^4,\ du = 36x^3\, dx] \\ &= \tfrac{2\pi}{36} \int_1^{145} \sqrt{u}\, du = \tfrac{\pi}{18} \left[\tfrac{2}{3} u^{3/2}\right]_1^{145} = \tfrac{\pi}{27}\left(145\sqrt{145} - 1\right) \end{aligned}$$

6. The curve $9x = y^2 + 18$ is symmetric about the x-axis, so we only use its top half, given by

$y = 3\sqrt{x - 2}$. $\dfrac{dy}{dx} = \dfrac{3}{2\sqrt{x - 2}}$, so $1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{9}{4(x - 2)}$. Thus,

$$\begin{aligned} S &= \int_2^6 2\pi \cdot 3\sqrt{x - 2} \sqrt{1 + \frac{9}{4(x - 2)}}\, dx = 6\pi \int_2^6 \sqrt{x - 2 + \tfrac{9}{4}}\, dx = 6\pi \int_2^6 \left(x + \tfrac{1}{4}\right)^{1/2} dx \\ &= 6\pi \cdot \tfrac{2}{3} \left[\left(x + \tfrac{1}{4}\right)^{3/2}\right]_2^6 = 4\pi \left[\left(\tfrac{25}{4}\right)^{3/2} - \left(\tfrac{9}{4}\right)^{3/2}\right] = 4\pi\left(\tfrac{125}{8} - \tfrac{27}{8}\right) = 4\pi \cdot \tfrac{98}{8} = 49\pi \end{aligned}$$

7. $y = \sqrt{1+4x} \;\Rightarrow\; y' = \frac{1}{2}(1+4x)^{-1/2}(4) = \dfrac{2}{\sqrt{1+4x}} \;\Rightarrow\; \sqrt{1+(y')^2} = \sqrt{1+\dfrac{4}{1+4x}} = \sqrt{\dfrac{5+4x}{1+4x}}$. So

$$S = \int_1^5 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi\int_1^5 \sqrt{1+4x}\sqrt{\frac{5+4x}{1+4x}}\,dx = 2\pi\int_1^5 \sqrt{4x+5}\,dx$$

$$= 2\pi\int_9^{25}\sqrt{u}\left(\tfrac{1}{4}\,du\right) \quad \begin{bmatrix} u = 4x+5, \\ du = 4\,dx \end{bmatrix} \quad = \tfrac{2\pi}{4}\left[\tfrac{2}{3}u^{3/2}\right]_9^{25} = \tfrac{\pi}{3}(25^{3/2} - 9^{3/2}) = \tfrac{\pi}{3}(125-27) = \tfrac{98}{3}\pi$$

8. $y = c + a\cosh(x/a) \;\Rightarrow\; y' = \sinh(x/a) \;\Rightarrow\; 1+(y')^2 = 1+\sinh^2(x/a) = \cosh^2(x/a) \;\Rightarrow$

$\sqrt{1+(y')^2} = \cosh(x/a)$. So

$$S = \int_0^a 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi\int_0^a \left[c + a\cosh\left(\frac{x}{a}\right)\right]\cosh\left(\frac{x}{a}\right)dx = 2\pi\int_0^a \left[c\cosh\left(\frac{x}{a}\right) + a\cosh^2\left(\frac{x}{a}\right)\right]dx$$

$$= 2\pi\int_0^a \left[c\cosh\left(\frac{x}{a}\right) + \frac{a}{2}\left(1+\cosh\left(\frac{2x}{a}\right)\right)\right]dx \qquad \left[\cosh^2 x = \tfrac{1}{2}(1+\cosh 2x)\right]$$

$$= 2\pi\left[ac\sinh\left(\frac{x}{a}\right) + \frac{ax}{2} + \frac{a^2}{4}\sinh\left(\frac{2x}{a}\right)\right]_0^a = 2\pi\left(ac\sinh 1 + \tfrac{1}{2}a^2 + \tfrac{1}{4}a^2\sinh 2\right)$$

9. $y = \sin\pi x \;\Rightarrow\; y' = \pi\cos\pi x \;\Rightarrow\; 1+(y')^2 = 1+\pi^2\cos^2(\pi x)$. So

$$S = \int_0^1 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi\int_0^1 \sin\pi x\sqrt{1+\pi^2\cos^2(\pi x)}\,dx \qquad \begin{bmatrix} u = \pi\cos\pi x, \\ du = -\pi^2\sin\pi x\,dx \end{bmatrix}$$

$$= 2\pi\int_\pi^{-\pi}\sqrt{1+u^2}\left(-\frac{1}{\pi^2}\,du\right) = \frac{2}{\pi}\int_{-\pi}^{\pi}\sqrt{1+u^2}\,du$$

$$= \frac{4}{\pi}\int_0^\pi \sqrt{1+u^2}\,du \overset{21}{=} \frac{4}{\pi}\left[\frac{u}{2}\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_0^\pi$$

$$= \frac{4}{\pi}\left[\left(\frac{\pi}{2}\sqrt{1+\pi^2} + \tfrac{1}{2}\ln\left(\pi+\sqrt{1+\pi^2}\right)\right) - 0\right] = 2\sqrt{1+\pi^2} + \frac{2}{\pi}\ln\left(\pi+\sqrt{1+\pi^2}\right)$$

10. $y = \dfrac{x^3}{6} + \dfrac{1}{2x} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{x^2}{2} - \dfrac{1}{2x^2} \;\Rightarrow\; \sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \sqrt{\dfrac{x^4}{4} + \dfrac{1}{2} + \dfrac{1}{4x^4}} = \sqrt{\left(\dfrac{x^2}{2} + \dfrac{1}{2x^2}\right)^2} = \dfrac{x^2}{2} + \dfrac{1}{2x^2} \;\Rightarrow$

$$S = \int_{1/2}^1 2\pi\left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)dx = 2\pi\int_{1/2}^1 \left(\frac{x^5}{12} + \frac{x}{12} + \frac{x}{4} + \frac{1}{4x^3}\right)dx$$

$$= 2\pi\int_{1/2}^1 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{x^{-3}}{4}\right)dx = 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{x^{-2}}{8}\right]_{1/2}^1$$

$$= 2\pi\left[\left(\tfrac{1}{72} + \tfrac{1}{6} - \tfrac{1}{8}\right) - \left(\tfrac{1}{64\cdot 72} + \tfrac{1}{24} - \tfrac{1}{2}\right)\right] = 2\pi\left(\tfrac{263}{512}\right) = \tfrac{263}{256}\pi$$

11. $x = \frac{1}{3}(y^2+2)^{3/2} \;\Rightarrow\; dx/dy = \frac{1}{2}(y^2+2)^{1/2}(2y) = y\sqrt{y^2+2} \;\Rightarrow\; 1+(dx/dy)^2 = 1+y^2(y^2+2) = (y^2+1)^2$.

So $S = 2\pi\int_1^2 y(y^2+1)\,dy = 2\pi\left[\frac{1}{4}y^4 + \frac{1}{2}y^2\right]_1^2 = 2\pi\left(4+2-\frac{1}{4}-\frac{1}{2}\right) = \frac{21\pi}{2}$.

12. $x = 1+2y^2 \;\Rightarrow\; 1+(dx/dy)^2 = 1+(4y)^2 = 1+16y^2$.

So $S = 2\pi\int_1^2 y\sqrt{1+16y^2}\,dy = \frac{\pi}{16}\int_1^2 (16y^2+1)^{1/2}32y\,dy = \frac{\pi}{16}\left[\frac{2}{3}(16y^2+1)^{3/2}\right]_1^2 = \frac{\pi}{24}\left(65\sqrt{65} - 17\sqrt{17}\right)$.

13. $y = \sqrt[3]{x} \;\Rightarrow\; x = y^3 \;\Rightarrow\; 1 + (dx/dy)^2 = 1 + 9y^4$. So

$$S = 2\pi \int_1^2 x\sqrt{1+(dx/dy)^2}\,dy = 2\pi \int_1^2 y^3\sqrt{1+9y^4}\,dy = \tfrac{2\pi}{36}\int_1^2 \sqrt{1+9y^4}\,36y^3\,dy = \tfrac{\pi}{18}\left[\tfrac{2}{3}\left(1+9y^4\right)^{3/2}\right]_1^2$$
$$= \tfrac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right)$$

14. $y = 1 - x^2 \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + 4x^2 \;\Rightarrow$

$$S = 2\pi \int_0^1 x\sqrt{1+4x^2}\,dx = \tfrac{\pi}{4}\int_0^1 8x\sqrt{4x^2+1}\,dx = \tfrac{\pi}{4}\left[\tfrac{2}{3}\left(4x^2+1\right)^{3/2}\right]_0^1 = \tfrac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

15. $x = \sqrt{a^2 - y^2} \;\Rightarrow\; dx/dy = \tfrac{1}{2}(a^2-y^2)^{-1/2}(-2y) = -y/\sqrt{a^2-y^2} \;\Rightarrow$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{a^2-y^2} = \frac{a^2-y^2}{a^2-y^2} + \frac{y^2}{a^2-y^2} = \frac{a^2}{a^2-y^2} \;\Rightarrow$$

$$S = \int_0^{a/2} 2\pi\sqrt{a^2-y^2}\frac{a}{\sqrt{a^2-y^2}}\,dy = 2\pi\int_0^{a/2} a\,dy = 2\pi a\big[y\big]_0^{a/2} = 2\pi a\left(\frac{a}{2} - 0\right) = \pi a^2.$$

Note that this is $\frac{1}{4}$ the surface area of a sphere of radius a, and the length of the interval $y = 0$ to $y = a/2$ is $\frac{1}{4}$ the length of the interval $y = -a$ to $y = a$.

16. $y = \tfrac{1}{4}x^2 - \tfrac{1}{2}\ln x \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{x}{2} - \dfrac{1}{2x} \;\Rightarrow\; 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{x^2}{4} - \dfrac{1}{2} + \dfrac{1}{4x^2} = \dfrac{x^2}{4} + \dfrac{1}{2} + \dfrac{1}{4x^2} = \left(\dfrac{x}{2} + \dfrac{1}{2x}\right)^2$. So

$$S = \int_1^2 2\pi x\sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}\,dx = 2\pi\int_1^2 x\left(\frac{x}{2}+\frac{1}{2x}\right)dx = \pi\int_1^2 (x^2+1)\,dx = \pi\left[\tfrac{1}{3}x^3 + x\right]_1^2$$
$$= \pi\left[\left(\tfrac{8}{3}+2\right) - \left(\tfrac{1}{3}+1\right)\right] = \tfrac{10}{3}\pi$$

17. $y = \ln x \;\Rightarrow\; dy/dx = 1/x \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + 1/x^2 \;\Rightarrow\; S = \int_1^3 2\pi \ln x\sqrt{1+1/x^2}\,dx$.

Let $f(x) = \ln x\sqrt{1+1/x^2}$. Since $n = 10$, $\Delta x = \frac{3-1}{10} = \frac{1}{5}$. Then

$S \approx S_{10} = 2\pi \cdot \frac{1/5}{3}\left[f(1) + 4f(1.2) + 2f(1.4) + \cdots + 2f(2.6) + 4f(2.8) + f(3)\right] \approx 9.023754$.

The value of the integral produced by a calculator is 9.024262 (to six decimal places).

18. $y = x + \sqrt{x} \;\Rightarrow\; dy/dx = 1 + \tfrac{1}{2}x^{-1/2} \;\Rightarrow\; 1 + (dy/dx)^2 = 2 + x^{-1/2} + \tfrac{1}{4}x^{-1} \;\Rightarrow$

$S = \displaystyle\int_1^2 2\pi\left(x+\sqrt{x}\right)\sqrt{2 + \frac{1}{\sqrt{x}} + \frac{1}{4x}}\,dx$. Let $f(x) = \left(x+\sqrt{x}\right)\sqrt{2+\dfrac{1}{\sqrt{x}}+\dfrac{1}{4x}}$. Since $n = 10$, $\Delta x = \frac{2-1}{10} = \frac{1}{10}$.

Then $S \approx S_{10} = 2\pi \cdot \frac{1/10}{3}\left[f(1) + 4f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 4f(1.9) + f(2)\right] \approx 29.506566$.

The value of the integral produced by a calculator is 29.506568 (to six decimal places).

19. $y = \sec x \;\Rightarrow\; dy/dx = \sec x\,\tan x \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + \sec^2 x\,\tan^2 x \;\Rightarrow$

$S = \int_0^{\pi/3} 2\pi\sec x\sqrt{1+\sec^2 x\,\tan^2 x}\,dx$. Let $f(x) = \sec x\sqrt{1+\sec^2 x\,\tan^2 x}$. Since $n = 10$, $\Delta x = \dfrac{\pi/3 - 0}{10} = \dfrac{\pi}{30}$.

Then $S \approx S_{10} = 2\pi \cdot \dfrac{\pi/30}{3}\left[f(0) + 4f\left(\dfrac{\pi}{30}\right) + 2f\left(\dfrac{2\pi}{30}\right) + \cdots + 2f\left(\dfrac{8\pi}{30}\right) + 4f\left(\dfrac{9\pi}{30}\right) + f\left(\dfrac{\pi}{3}\right)\right] \approx 13.527296$.

The value of the integral produced by a calculator is 13.516987 (to six decimal places).

20. $y = e^{-x^2} \;\Rightarrow\; dy/dx = -2xe^{-x^2} \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + 4x^2e^{-2x^2} \;\Rightarrow$

$S = \int_0^1 2\pi\, e^{-x^2}\sqrt{1 + 4x^2e^{-2x^2}}\, dx$. Let $f(x) = e^{-x^2}\sqrt{1 + 4x^2e^{-2x^2}}$. Since $n = 10$, $\Delta x = \frac{1-0}{10} = \frac{1}{10}$.

Then $S \approx S_{10} = 2\pi \cdot \frac{1/10}{3}\left[f(0) + 4f\left(\frac{1}{10}\right) + 2f\left(\frac{2}{10}\right) + \cdots + 2f\left(\frac{8}{10}\right) + 4f\left(\frac{9}{10}\right) + f(1)\right] \approx 5.537658$.

The value of the integral produced by a calculator is 5.537643 (to six decimal places).

21. $y = 1/x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\, dx = \sqrt{1 + (-1/x^2)^2}\, dx = \sqrt{1 + 1/x^4}\, dx \;\Rightarrow$

$$S = \int_1^2 2\pi \cdot \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx = 2\pi \int_1^2 \frac{\sqrt{x^4 + 1}}{x^3}\, dx = 2\pi \int_1^4 \frac{\sqrt{u^2 + 1}}{u^2}\left(\tfrac{1}{2}\, du\right) \qquad [u = x^2,\, du = 2x\, dx]$$

$$= \pi \int_1^4 \frac{\sqrt{1 + u^2}}{u^2}\, du \overset{24}{=} \pi\left[-\frac{\sqrt{1 + u^2}}{u} + \ln\left(u + \sqrt{1 + u^2}\right)\right]_1^4$$

$$= \pi\left[-\tfrac{\sqrt{17}}{4} + \ln\left(4 + \sqrt{17}\right) + \tfrac{\sqrt{2}}{1} - \ln\left(1 + \sqrt{2}\right)\right] = \frac{\pi}{4}\left[4\ln\left(\sqrt{17} + 4\right) - 4\ln\left(\sqrt{2} + 1\right) - \sqrt{17} + 4\sqrt{2}\right]$$

22. $y = \sqrt{x^2 + 1} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2 + 1}} \;\Rightarrow\; ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \dfrac{x^2}{x^2 + 1}}\, dx \;\Rightarrow$

$$S = \int_0^3 2\pi\sqrt{x^2 + 1}\sqrt{1 + \frac{x^2}{x^2 + 1}}\, dx = 2\pi \int_0^3 \sqrt{2x^2 + 1}\, dx = 2\sqrt{2}\,\pi \int_0^3 \sqrt{x^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2}\, dx$$

$$\overset{21}{=} 2\sqrt{2}\pi\left[\tfrac{1}{2}x\sqrt{x^2 + \tfrac{1}{2}} + \tfrac{1}{4}\ln\left(x + \sqrt{x^2 + \tfrac{1}{2}}\right)\right]_0^3 = 2\sqrt{2}\pi\left[\tfrac{3}{2}\sqrt{9 + \tfrac{1}{2}} + \tfrac{1}{4}\ln\left(3 + \sqrt{9 + \tfrac{1}{2}}\right) - \tfrac{1}{4}\ln\tfrac{1}{\sqrt{2}}\right]$$

$$= 2\sqrt{2}\pi\left[\tfrac{3}{2}\sqrt{\tfrac{19}{2}} + \tfrac{1}{4}\ln\left(3 + \sqrt{\tfrac{19}{2}}\right) + \tfrac{1}{4}\ln\sqrt{2}\right] = 2\sqrt{2}\pi\left[\tfrac{3}{2}\tfrac{\sqrt{19}}{\sqrt{2}} + \tfrac{1}{4}\ln\left(3\sqrt{2} + \sqrt{19}\right)\right]$$

$$= 3\sqrt{19}\pi + \tfrac{\pi}{\sqrt{2}}\ln\left(3\sqrt{2} + \sqrt{19}\right)$$

23. $y = x^3$ and $0 \le y \le 1 \;\Rightarrow\; y' = 3x^2$ and $0 \le x \le 1$.

$$S = \int_0^1 2\pi x\sqrt{1 + (3x^2)^2}\, dx = 2\pi \int_0^3 \sqrt{1 + u^2}\,\tfrac{1}{6}\, du \quad \begin{bmatrix} u = 3x^2, \\ du = 6x\, dx \end{bmatrix} \quad = \tfrac{\pi}{3}\int_0^3 \sqrt{1 + u^2}\, du$$

$$\overset{21}{=} \text{[or use CAS]} \;\; \tfrac{\pi}{3}\left[\tfrac{1}{2}u\sqrt{1 + u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1 + u^2}\right)\right]_0^3 = \tfrac{\pi}{3}\left[\tfrac{3}{2}\sqrt{10} + \tfrac{1}{2}\ln\left(3 + \sqrt{10}\right)\right] = \tfrac{\pi}{6}\left[3\sqrt{10} + \ln\left(3 + \sqrt{10}\right)\right]$$

24. $y = \ln(x + 1)$, $0 \le x \le 1$. $ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\dfrac{1}{x + 1}\right)^2}\, dx$, so

$$S = \int_0^1 2\pi x\sqrt{1 + \frac{1}{(x + 1)^2}}\, dx = \int_1^2 2\pi(u - 1)\sqrt{1 + \frac{1}{u^2}}\, du \qquad [u = x + 1,\, du = dx]$$

$$= 2\pi \int_1^2 u\,\frac{\sqrt{1 + u^2}}{u}\, du - 2\pi \int_1^2 \frac{\sqrt{1 + u^2}}{u}\, du = 2\pi \int_1^2 \sqrt{1 + u^2}\, du - 2\pi \int_1^2 \frac{\sqrt{1 + u^2}}{u}\, du$$

$$\overset{21,\,23}{=} \text{[or use CAS]} \;\; 2\pi\left[\tfrac{1}{2}u\sqrt{1 + u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1 + u^2}\right)\right]_1^2 - 2\pi\left[\sqrt{1 + u^2} - \ln\left(\frac{1 + \sqrt{1 + u^2}}{u}\right)\right]_1^2$$

$$= 2\pi\left[\sqrt{5} + \tfrac{1}{2}\ln\left(2 + \sqrt{5}\right) - \tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\ln\left(1 + \sqrt{2}\right)\right] - 2\pi\left[\sqrt{5} - \ln\left(\tfrac{1 + \sqrt{5}}{2}\right) - \sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$$

$$= 2\pi\left[\tfrac{1}{2}\ln\left(2 + \sqrt{5}\right) + \ln\left(\tfrac{1 + \sqrt{5}}{2}\right) + \tfrac{\sqrt{2}}{2} - \tfrac{3}{2}\ln\left(1 + \sqrt{2}\right)\right]$$

25. $S = 2\pi \int_1^\infty y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx$. Rather than trying to evaluate this integral, note that $\sqrt{x^4+1} > \sqrt{x^4} = x^2$ for $x > 0$. Thus, if the area is finite,

$S = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx > 2\pi \int_1^\infty \frac{x^2}{x^3}\, dx = 2\pi \int_1^\infty \frac{1}{x}\, dx$. But we know that this integral diverges, so the area S is infinite.

26. $S = \int_0^\infty 2\pi y \sqrt{1 + (dy/dx)^2}\, dx = 2\pi \int_0^\infty e^{-x}\sqrt{1 + (-e^{-x})^2}\, dx \qquad [y = e^{-x},\, y' = -e^{-x}]$.

Evaluate $I = \int e^{-x}\sqrt{1 + (-e^{-x})^2}\, dx$ by using the substitution $u = -e^{-x}$, $du = e^{-x}\, dx$:

$I = \int \sqrt{1+u^2}\, du \overset{21}{=} \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln\left(u + \sqrt{1+u^2}\right) + C = \frac{1}{2}(-e^{-x})\sqrt{1+e^{-2x}} + \frac{1}{2}\ln\left(-e^{-x} + \sqrt{1+e^{-2x}}\right) + C.$

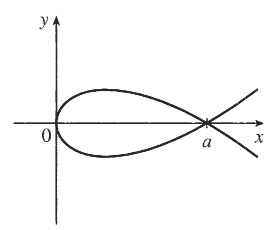
Returning to the surface area integral, we have

$$\begin{aligned}
S &= 2\pi \lim_{t\to\infty} \int_0^t e^{-x}\sqrt{1+(-e^{-x})^2}\, dx = 2\pi \lim_{t\to\infty} \left[\tfrac{1}{2}(-e^{-x})\sqrt{1+e^{-2x}} + \tfrac{1}{2}\ln\left(-e^{-x} + \sqrt{1+e^{-2x}}\right)\right]_0^t \\
&= 2\pi \lim_{t\to\infty} \left\{\left[\tfrac{1}{2}(-e^{-t})\sqrt{1+e^{-2t}} + \tfrac{1}{2}\ln\left(-e^{-t} + \sqrt{1+e^{-2t}}\right)\right] - \left[\tfrac{1}{2}(-1)\sqrt{1+1} + \tfrac{1}{2}\ln\left(-1 + \sqrt{1+1}\right)\right]\right\} \\
&= 2\pi \left\{\left[\tfrac{1}{2}(0)\sqrt{1} + \tfrac{1}{2}\ln\left(0 + \sqrt{1}\right)\right] - \left[-\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\ln\left(-1 + \sqrt{2}\right)\right]\right\} \\
&= 2\pi\left\{[0] + \tfrac{1}{2}\left[\sqrt{2} - \ln\left(\sqrt{2}-1\right)\right]\right\} = \pi\left[\sqrt{2} - \ln\left(\sqrt{2}-1\right)\right]
\end{aligned}$$

27. Since $a > 0$, the curve $3ay^2 = x(a-x)^2$ only has points with $x \geq 0$.

$[3ay^2 \geq 0 \;\Rightarrow\; x(a-x)^2 \geq 0 \;\Rightarrow\; x \geq 0.]$

The curve is symmetric about the x-axis (since the equation is unchanged when y is replaced by $-y$). $y = 0$ when $x = 0$ or a, so the curve's loop extends from $x = 0$ to $x = a$.



$\frac{d}{dx}(3ay^2) = \frac{d}{dx}[x(a-x)^2] \;\Rightarrow\; 6ay\frac{dy}{dx} = x\cdot 2(a-x)(-1) + (a-x)^2 \;\Rightarrow\; \frac{dy}{dx} = \frac{(a-x)[-2x+a-x]}{6ay} \;\Rightarrow$

$\left(\frac{dy}{dx}\right)^2 = \frac{(a-x)^2(a-3x)^2}{36a^2y^2} = \frac{(a-x)^2(a-3x)^2}{36a^2}\cdot\frac{3a}{x(a-x)^2} \quad \left[\begin{array}{c}\text{the last fraction}\\ \text{is } 1/y^2\end{array}\right] = \frac{(a-3x)^2}{12ax} \;\Rightarrow$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{a^2 - 6ax + 9x^2}{12ax} = \frac{12ax}{12ax} + \frac{a^2 - 6ax + 9x^2}{12ax} = \frac{a^2 + 6ax + 9x^2}{12ax} = \frac{(a+3x)^2}{12ax}$ for $x \neq 0$.

(a) $S = \int_{x=0}^a 2\pi y\, ds = 2\pi \int_0^a \frac{\sqrt{x}\,(a-x)}{\sqrt{3a}}\cdot\frac{a+3x}{\sqrt{12ax}}\, dx = 2\pi \int_0^a \frac{(a-x)(a+3x)}{6a}\, dx$

$= \frac{\pi}{3a}\int_0^a (a^2 + 2ax - 3x^2)\, dx = \frac{\pi}{3a}\left[a^2x + ax^2 - x^3\right]_0^a = \frac{\pi}{3a}(a^3 + a^3 - a^3) = \frac{\pi}{3a}\cdot a^3 = \frac{\pi a^2}{3}.$

Note that we have rotated the top half of the loop about the x-axis. This generates the full surface.

(b) We must rotate the full loop about the y-axis, so we get double the area obtained by rotating the top half of the loop:

$$\begin{aligned}
S &= 2\cdot 2\pi \int_{x=0}^a x\, ds = 4\pi \int_0^a x\frac{a+3x}{\sqrt{12ax}}\, dx = \frac{4\pi}{2\sqrt{3a}}\int_0^a x^{1/2}(a+3x)\, dx = \frac{2\pi}{\sqrt{3a}}\int_0^a (ax^{1/2} + 3x^{3/2})\, dx \\
&= \frac{2\pi}{\sqrt{3a}}\left[\frac{2}{3}ax^{3/2} + \frac{6}{5}x^{5/2}\right]_0^a = \frac{2\pi\sqrt{3}}{3\sqrt{a}}\left(\frac{2}{3}a^{5/2} + \frac{6}{5}a^{5/2}\right) = \frac{2\pi\sqrt{3}}{3}\left(\frac{2}{3} + \frac{6}{5}\right)a^2 = \frac{2\pi\sqrt{3}}{3}\left(\frac{28}{15}\right)a^2 \\
&= \frac{56\pi\sqrt{3}\,a^2}{45}
\end{aligned}$$

28. In general, if the parabola $y = ax^2$, $-c \le x \le c$, is rotated about the y-axis, the surface area it generates is

$$2\pi \int_0^c x\sqrt{1+(2ax)^2}\,dx = 2\pi \int_0^{2ac} \frac{u}{2a}\sqrt{1+u^2}\,\frac{1}{2a}\,du \quad \begin{bmatrix} u = 2ax, \\ du = 2a\,dx \end{bmatrix} \quad = \frac{\pi}{4a^2}\int_0^{2ac} \left(1+u^2\right)^{1/2} 2u\,du$$

$$= \frac{\pi}{4a^2}\left[\tfrac{2}{3}\left(1+u^2\right)^{3/2}\right]_0^{2ac} = \frac{\pi}{6a^2}\left[\left(1+4a^2c^2\right)^{3/2} - 1\right]$$

Here $2c = 10$ ft and $ac^2 = 2$ ft, so $c = 5$ and $a = \frac{2}{25}$. Thus, the surface area is

$$S = \tfrac{\pi}{6}\tfrac{625}{4}\left[\left(1 + 4\cdot\tfrac{4}{625}\cdot 25\right)^{3/2} - 1\right] = \tfrac{625\pi}{24}\left[\left(1+\tfrac{16}{25}\right)^{3/2} - 1\right] = \tfrac{625\pi}{24}\left(\tfrac{41\sqrt{41}}{125} - 1\right) = \tfrac{5\pi}{24}\left(41\sqrt{41} - 125\right) \approx 90.01 \text{ ft}^2.$$

29. (a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{y\,(dy/dx)}{b^2} = -\dfrac{x}{a^2} \;\Rightarrow\; \dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y} \;\Rightarrow$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{b^4x^2}{a^4y^2} = \frac{b^4x^2 + a^4y^2}{a^4y^2} = \frac{b^4x^2 + a^4b^2\left(1 - x^2/a^2\right)}{a^4b^2\left(1 - x^2/a^2\right)} = \frac{a^4b^2 + b^4x^2 - a^2b^2x^2}{a^4b^2 - a^2b^2x^2}$$

$$= \frac{a^4 + b^2x^2 - a^2x^2}{a^4 - a^2x^2} = \frac{a^4 - \left(a^2 - b^2\right)x^2}{a^2\left(a^2 - x^2\right)}$$

The ellipsoid's surface area is twice the area generated by rotating the first-quadrant portion of the ellipse about the x-axis. Thus,

$$S = 2\int_0^a 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 4\pi\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,\frac{\sqrt{a^4 - (a^2 - b^2)x^2}}{a\sqrt{a^2 - x^2}}\,dx = \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4 - (a^2 - b^2)x^2}\,dx$$

$$= \frac{4\pi b}{a^2}\int_0^{a\sqrt{a^2-b^2}} \sqrt{a^4 - u^2}\,\frac{du}{\sqrt{a^2 - b^2}} \quad \left[u = \sqrt{a^2 - b^2}\,x\right] \quad \overset{30}{=} \frac{4\pi b}{a^2\sqrt{a^2 - b^2}}\left[\frac{u}{2}\sqrt{a^4 - u^2} + \frac{a^4}{2}\sin^{-1}\left(\frac{u}{a^2}\right)\right]_0^{a\sqrt{a^2-b^2}}$$

$$= \frac{4\pi b}{a^2\sqrt{a^2 - b^2}}\left[\frac{a\sqrt{a^2 - b^2}}{2}\sqrt{a^4 - a^2(a^2 - b^2)} + \frac{a^4}{2}\sin^{-1}\frac{\sqrt{a^2 - b^2}}{a}\right] = 2\pi\left[b^2 + \frac{a^2b\sin^{-1}\dfrac{\sqrt{a^2 - b^2}}{a}}{\sqrt{a^2 - b^2}}\right]$$

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{x\,(dx/dy)}{a^2} = -\dfrac{y}{b^2} \;\Rightarrow\; \dfrac{dx}{dy} = -\dfrac{a^2y}{b^2x} \;\Rightarrow$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{a^4y^2}{b^4x^2} = \frac{b^4x^2 + a^4y^2}{b^4x^2} = \frac{b^4a^2(1 - y^2/b^2) + a^4y^2}{b^4a^2(1 - y^2/b^2)} = \frac{a^2b^4 - a^2b^2y^2 + a^4y^2}{a^2b^4 - a^2b^2y^2}$$

$$= \frac{b^4 - b^2y^2 + a^2y^2}{b^4 - b^2y^2} = \frac{b^4 - (b^2 - a^2)y^2}{b^2(b^2 - y^2)}$$

The oblate spheroid's surface area is twice the area generated by rotating the first-quadrant portion of the ellipse about the y-axis. Thus,

$$S = 2\int_0^b 2\pi\,x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = 4\pi\int_0^b \frac{a}{b}\sqrt{b^2 - y^2}\,\frac{\sqrt{b^4 - (b^2 - a^2)y^2}}{b\sqrt{b^2 - y^2}}\,dy$$

$$= \frac{4\pi a}{b^2}\int_0^b \sqrt{b^4 - (b^2 - a^2)y^2}\,dy = \frac{4\pi a}{b^2}\int_0^{b\sqrt{b^2-a^2}} \sqrt{b^4 - u^2}\,\frac{du}{\sqrt{b^2 - a^2}} \qquad \left[u = \sqrt{b^2 - a^2}\,y\right]$$

$$\overset{30}{=} \frac{4\pi a}{b^2\sqrt{b^2 - a^2}}\left[\frac{u}{2}\sqrt{b^4 - u^2} + \frac{b^4}{2}\sin^{-1}\left(\frac{u}{b^2}\right)\right]_0^{b\sqrt{b^2-a^2}}$$

$$= \frac{4\pi a}{b^2\sqrt{b^2 - a^2}}\left[\frac{b\sqrt{b^2 - a^2}}{2}\sqrt{b^4 - b^2(b^2 - a^2)} + \frac{b^4}{2}\sin^{-1}\frac{\sqrt{b^2 - a^2}}{b}\right] = 2\pi\left[a^2 + \frac{ab^2\sin^{-1}\dfrac{\sqrt{b^2 - a^2}}{b}}{\sqrt{b^2 - a^2}}\right]$$

Notice that this result can be obtained from the answer in part (a) by interchanging a and b.

30. The upper half of the torus is generated by rotating the curve $(x-R)^2+y^2=r^2$, $y>0$, about the y-axis.

$y\,\dfrac{dy}{dx}=-(x-R)\;\Rightarrow\;1+\left(\dfrac{dy}{dx}\right)^2=1+\dfrac{(x-R)^2}{y^2}=\dfrac{y^2+(x-R)^2}{y^2}=\dfrac{r^2}{r^2-(x-R)^2}$. Thus,

$$S=2\int_{R-r}^{R+r}2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=4\pi\int_{R-r}^{R+r}\frac{rx}{\sqrt{r^2-(x-R)^2}}\,dx=4\pi r\int_{-r}^{r}\frac{u+R}{\sqrt{r^2-u^2}}\,du\qquad[u=x-R]$$

$$=4\pi r\int_{-r}^{r}\frac{u\,du}{\sqrt{r^2-u^2}}+4\pi Rr\int_{-r}^{r}\frac{du}{\sqrt{r^2-u^2}}=4\pi r\cdot 0+8\pi Rr\int_{0}^{r}\frac{du}{\sqrt{r^2-u^2}}\qquad\left[\begin{array}{c}\text{since the first integrand is odd}\\ \text{and the second is even}\end{array}\right]$$

$$=8\pi Rr\left[\sin^{-1}(u/r)\right]_0^r=8\pi Rr\left(\tfrac{\pi}{2}\right)=4\pi^2Rr$$

31. The analogue of $f(x_i^*)$ in the derivation of (4) is now $c-f(x_i^*)$, so

$$S=\lim_{n\to\infty}\sum_{i=1}^{n}2\pi[c-f(x_i^*)]\sqrt{1+[f'(x_i^*)]^2}\,\Delta x=\int_a^b 2\pi[c-f(x)]\sqrt{1+[f'(x)]^2}\,dx.$$

32. $y=x^{1/2}\;\Rightarrow\;y'=\frac{1}{2}x^{-1/2}\;\Rightarrow\;1+(y')^2=1+1/4x$, so by Exercise 31, $S=\int_0^4 2\pi\left(4-\sqrt{x}\right)\sqrt{1+1/(4x)}\,dx$.

Using a CAS, we get $S=2\pi\ln\left(\sqrt{17}+4\right)+\frac{\pi}{6}\left(31\sqrt{17}+1\right)\approx 80.6095$.

33. For the upper semicircle, $f(x)=\sqrt{r^2-x^2}$, $f'(x)=-x/\sqrt{r^2-x^2}$. The surface area generated is

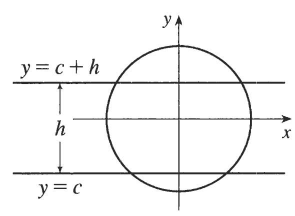
$$S_1=\int_{-r}^{r}2\pi\left(r-\sqrt{r^2-x^2}\right)\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx=4\pi\int_0^r\left(r-\sqrt{r^2-x^2}\right)\frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$=4\pi\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}-r\right)dx$$

For the lower semicircle, $f(x)=-\sqrt{r^2-x^2}$ and $f'(x)=\dfrac{x}{\sqrt{r^2-x^2}}$, so $S_2=4\pi\displaystyle\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}+r\right)dx$.

Thus, the total area is $S=S_1+S_2=8\pi\displaystyle\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}\right)dx=8\pi\left[r^2\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r=8\pi r^2\left(\frac{\pi}{2}\right)=4\pi^2r^2$.

34. Take the sphere $x^2+y^2+z^2=\frac{1}{4}d^2$ and let the intersecting planes be $y=c$ and $y=c+h$, where $-\frac{1}{2}d\le c\le\frac{1}{2}d-h$. The sphere intersects the xy-plane in the circle $x^2+y^2=\frac{1}{4}d^2$. From this equation, we get $x\,\dfrac{dx}{dy}+y=0$, so $\dfrac{dx}{dy}=-\dfrac{y}{x}$. The desired surface area is



$$S=2\pi\int x\,ds=2\pi\int_c^{c+h}x\sqrt{1+(dx/dy)^2}\,dy=2\pi\int_c^{c+h}x\sqrt{1+y^2/x^2}\,dy=2\pi\int_c^{c+h}\sqrt{x^2+y^2}\,dy$$

$$=2\pi\int_c^{c+h}\tfrac{1}{2}d\,dy=\pi d\int_c^{c+h}dy=\pi dh$$

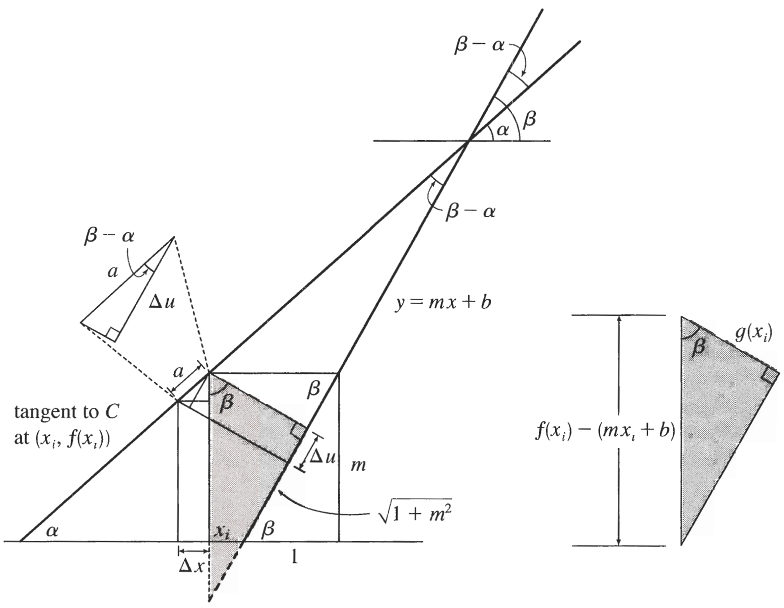
35. In the derivation of (4), we computed a typical contribution to the surface area to be $2\pi\,\dfrac{y_{i-1}+y_i}{2}\,|P_{i-1}P_i|$, the area of a frustum of a cone. When $f(x)$ is not necessarily positive, the approximations $y_i=f(x_i)\approx f(x_i^*)$ and $y_{i-1}=f(x_{i-1})\approx f(x_i^*)$ must be replaced by $y_i=|f(x_i)|\approx|f(x_i^*)|$ and $y_{i-1}=|f(x_{i-1})|\approx|f(x_i^*)|$. Thus, $2\pi\,\dfrac{y_{i-1}+y_i}{2}\,|P_{i-1}P_i|\approx 2\pi\,|f(x_i^*)|\sqrt{1+[f'(x_i^*)]^2}\,\Delta x$. Continuing with the rest of the derivation as before, we obtain $S=\int_a^b 2\pi\,|f(x)|\sqrt{1+[f'(x)]^2}\,dx$.

36. Since $g(x) = f(x) + c$, we have $g'(x) = f'(x)$. Thus,

$$\begin{aligned} S_g &= \textstyle\int_a^b 2\pi g(x)\sqrt{1+[g'(x)]^2}\,dx = \int_a^b 2\pi[f(x)+c]\sqrt{1+[f'(x)]^2}\,dx \\ &= \textstyle\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx + 2\pi c\int_a^b \sqrt{1+[f'(x)]^2}\,dx = S_f + 2\pi cL \end{aligned}$$

DISCOVERY PROJECT Rotating on a Slant

1.



In the figure, the segment a lying above the interval $[x_i - \Delta x, x_i]$ along the tangent to C has length $\Delta x \sec\alpha = \Delta x\sqrt{1+\tan^2\alpha} = \sqrt{1+[f'(x_i)]^2}\,\Delta x$. The segment from $(x_i, f(x_i))$ drawn perpendicular to the line $y = mx + b$ has length

$$g(x_i) = [f(x_i) - mx_i - b]\cos\beta = \frac{f(x_i) - mx_i - b}{\sec\beta} = \frac{f(x_i) - mx_i - b}{\sqrt{1+\tan^2\beta}} = \frac{f(x_i) - mx_i - b}{\sqrt{1+m^2}}$$

Also, $\cos(\beta - \alpha) = \dfrac{\Delta u}{\Delta x\ \sec\alpha} \quad \Rightarrow$

$$\begin{aligned} \Delta u &= \Delta x\ \sec\alpha\ \cos(\beta-\alpha) = \Delta x\,\frac{\cos\beta\ \cos\alpha + \sin\beta\ \sin\alpha}{\cos\alpha} = \Delta x(\cos\beta + \sin\beta\ \tan\alpha) \\ &= \Delta x\left[\frac{1}{\sqrt{1+m^2}} + \frac{m}{\sqrt{1+m^2}}f'(x_i)\right] = \frac{1+mf'(x_i)}{\sqrt{1+m^2}}\,\Delta x \end{aligned}$$

Thus,

$$\begin{aligned} \text{Area}(\mathfrak{R}) &= \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\,\Delta u = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{f(x_i) - mx_i - b}{\sqrt{1+m^2}}\cdot\frac{1+mf'(x_i)}{\sqrt{1+m^2}}\,\Delta x \\ &= \frac{1}{1+m^2}\int_p^q [f(x) - mx - b][1+mf'(x)]\,dx \end{aligned}$$

2. From Problem 1 with $m = 1$, $f(x) = x + \sin x$, $mx + b = x - 2$, $p = 0$, and $q = 2\pi$,

$$\text{Area} = \frac{1}{1 + 1^2} \int_0^{2\pi} [x + \sin x - (x - 2)]\,[1 + 1(1 + \cos x)]\,dx = \tfrac{1}{2} \int_0^{2\pi} (\sin x + 2)(2 + \cos x)\,dx$$

$$= \tfrac{1}{2} \int_0^{2\pi} (2\sin x + \sin x\,\cos x + 4 + 2\cos x)\,dx = \tfrac{1}{2}\left[-2\cos x + \tfrac{1}{2}\sin^2 x + 4x + 2\sin x\right]_0^{2\pi}$$

$$= \tfrac{1}{2}[(-2 + 0 + 8\pi + 0) - (-2 + 0 + 0 + 0)] = \tfrac{1}{2}(8\pi) = 4\pi$$

3. $$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi[g(x_i)]^2\,\Delta u = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left[\frac{f(x_i) - mx_i - b}{\sqrt{1 + m^2}}\right]^2 \frac{1 + mf'(x_i)}{\sqrt{1 + m^2}}\,\Delta x$$

$$= \frac{\pi}{(1 + m^2)^{3/2}} \int_p^q [f(x) - mx - b]^2 [1 + mf'(x)]\,dx$$

4. $$V = \frac{\pi}{(1 + 1^2)^{3/2}} \int_0^{2\pi} (x + \sin x - x + 2)^2 (1 + 1 + \cos x)\,dx$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{2\pi} (\sin x + 2)^2 (\cos x + 2)\,dx = \frac{\pi}{2\sqrt{2}} \int_0^{2\pi} \left(\sin^2 x + 4\sin x + 4\right)(\cos x + 2)\,dx$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{2\pi} \left(\sin^2 x\,\cos x + 4\sin x\,\cos x + 4\cos x + 2\sin^2 x + 8\sin x + 8\right) dx$$

$$= \frac{\pi}{2\sqrt{2}} \left[\tfrac{1}{3}\sin^3 x + 2\sin^2 x + 4\sin x + x - \tfrac{1}{2}\sin 2x - 8\cos x + 8x\right]_0^{2\pi} \qquad [\text{since } 2\sin^2 x = 1 - \cos 2x]$$

$$= \frac{\pi}{2\sqrt{2}} [(2\pi - 8 + 16\pi) - (-8)] = \frac{9\sqrt{2}}{2}\pi^2$$

5. $$S = \int_p^q 2\pi g(x)\sqrt{1 + [f'(x)]^2}\,dx = \frac{2\pi}{\sqrt{1 + m^2}} \int_p^q [f(x) - mx - b]\sqrt{1 + [f'(x)]^2}\,dx$$

6. From Problem 5 with $f(x) = \sqrt{x}$, $p = 0$, $q = 4$, $m = \frac{1}{2}$, and $b = 0$,

$$S = \frac{2\pi}{\sqrt{1 + (\frac{1}{2})^2}} \int_0^4 \left(\sqrt{x} - \tfrac{1}{2}x\right)\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\,dx \overset{\text{CAS}}{=} \frac{\pi}{\sqrt{5}}\left[\frac{\ln(\sqrt{17} + 4)}{32} + \frac{37\sqrt{17}}{24} - \frac{1}{3}\right] \approx 8.554$$