8 Review

CONCEPT CHECK

1. (a) The length of a curve is defined to be the limit of the lengths of the inscribed polygons, as described near Figure 3 in Section 8.1.

(b) See Equation 8.1.2.

(c) See Equation 8.1.4.

2. (a) $S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$

(b) If $x = g(y)$, $c \le y \le d$, then $S = \int_c^d 2\pi y\,\sqrt{1 + [g'(y)]^2}\,dy$.

(c) $S = \int_a^b 2\pi x\,\sqrt{1 + [f'(x)]^2}\,dx$ or $S = \int_c^d 2\pi g(y)\,\sqrt{1 + [g'(y)]^2}\,dy$

3. Let $c(x)$ be the cross-sectional length of the wall (measured parallel to the surface of the fluid) at depth x. Then the hydrostatic force against the wall is given by $F = \int_a^b \delta x c(x)\,dx$, where a and b are the lower and upper limits for x at points of the wall and δ is the weight density of the fluid.

4. (a) The center of mass is the point at which the plate balances horizontally.

(b) See Equations 8.3.8.

5. If a plane region $\mathcal{R}$ that lies entirely on one side of a line ℓ in its plane is rotated about ℓ, then the volume of the resulting solid is the product of the area of $\mathcal{R}$ and the distance traveled by the centroid of $\mathcal{R}$.

6. See Figure 3 in Section 8.4, and the discussion which precedes it.

7. (a) See the definition in the first paragraph of the subsection *Cardiac Output* in Section 8.4.

(b) See the discussion in the second paragraph of the subsection *Cardiac Output* in Section 8.4.

8. A probability density function f is a function on the domain of a continuous random variable X such that $\int_a^b f(x)\,dx$ measures the probability that X lies between a and b. Such a function f has nonnegative values and satisfies the relation $\int_D f(x)\,dx = 1$, where D is the domain of the corresponding random variable X. If $D = \mathbb{R}$, or if we define $f(x) = 0$ for real numbers $x \notin D$, then $\int_{-\infty}^{\infty} f(x)\,dx = 1$. (Of course, to work with f in this way, we must assume that the integrals of f exist.)

9. (a) $\int_0^{130} f(x)\,dx$ represents the probability that the weight of a randomly chosen female college student is less than 130 pounds.

(b) $\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x f(x)\,dx$

(c) The median of f is the number m such that $\int_m^{\infty} f(x)\,dx = \frac{1}{2}$.

10. See the discussion near Equation 3 in Section 8.5.

EXERCISES

1. $y = \frac{1}{6}(x^2+4)^{3/2} \Rightarrow dy/dx = \frac{1}{4}(x^2+4)^{1/2}(2x) \Rightarrow$

$1 + (dy/dx)^2 = 1 + \left[\frac{1}{2}x(x^2+4)^{1/2}\right]^2 = 1 + \frac{1}{4}x^2(x^2+4) = \frac{1}{4}x^4 + x^2 + 1 = \left(\frac{1}{2}x^2+1\right)^2.$

Thus, $L = \int_0^3 \sqrt{\left(\frac{1}{2}x^2+1\right)^2}\,dx = \int_0^3 \left(\frac{1}{2}x^2+1\right)dx = \left[\frac{1}{6}x^3 + x\right]_0^3 = \frac{15}{2}.$

2. $y = 2\ln\left(\sin\frac{1}{2}x\right) \Rightarrow \dfrac{dy}{dx} = 2\cdot\dfrac{1}{\sin\left(\frac{1}{2}x\right)}\cdot\cos\left(\frac{1}{2}x\right)\cdot\frac{1}{2} = \cot\left(\frac{1}{2}x\right) \Rightarrow 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \cot^2\left(\frac{1}{2}x\right) = \csc^2\left(\frac{1}{2}x\right).$

Thus,

$$L = \int_{\pi/3}^{\pi} \sqrt{\csc^2\left(\tfrac{1}{2}x\right)}\,dx = \int_{\pi/3}^{\pi} \left|\csc\left(\tfrac{1}{2}x\right)\right| dx = \int_{\pi/3}^{\pi} \csc\left(\tfrac{1}{2}x\right) dx = \int_{\pi/6}^{\pi/2} \csc u\,(2\,du) \quad \begin{bmatrix} u = \frac{1}{2}x, \\ du = \frac{1}{2}\,dx \end{bmatrix}$$

$$= 2\Big[\ln|\csc u - \cot u|\Big]_{\pi/6}^{\pi/2} = 2\left[\ln\left|\csc\tfrac{\pi}{2} - \cot\tfrac{\pi}{2}\right| - \ln\left|\csc\tfrac{\pi}{6} - \cot\tfrac{\pi}{6}\right|\right]$$

$$= 2\Big[\ln|1-0| - \ln\big|2-\sqrt{3}\big|\Big] = -2\ln\left(2-\sqrt{3}\right) \approx 2.63$$

3. (a) $y = \dfrac{x^4}{16} + \dfrac{1}{2x^2} = \dfrac{1}{16}x^4 + \dfrac{1}{2}x^{-2} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{4}x^3 - x^{-3} \Rightarrow$

$1 + (dy/dx)^2 = 1 + \left(\frac{1}{4}x^3 - x^{-3}\right)^2 = 1 + \frac{1}{16}x^6 - \frac{1}{2} + x^{-6} = \frac{1}{16}x^6 + \frac{1}{2} + x^{-6} = \left(\frac{1}{4}x^3 + x^{-3}\right)^2.$

Thus, $L = \int_1^2 \left(\frac{1}{4}x^3 + x^{-3}\right)dx = \left[\frac{1}{16}x^4 - \frac{1}{2}x^{-2}\right]_1^2 = \left(1 - \frac{1}{8}\right) - \left(\frac{1}{16} - \frac{1}{2}\right) = \frac{21}{16}.$

(b) $S = \int_1^2 2\pi x\left(\frac{1}{4}x^3 + x^{-3}\right)dx = 2\pi\int_1^2 \left(\frac{1}{4}x^4 + x^{-2}\right)dx = 2\pi\left[\frac{1}{20}x^5 - \frac{1}{x}\right]_1^2$

$= 2\pi\left[\left(\frac{32}{20} - \frac{1}{2}\right) - \left(\frac{1}{20} - 1\right)\right] = 2\pi\left(\frac{8}{5} - \frac{1}{2} - \frac{1}{20} + 1\right) = 2\pi\left(\frac{41}{20}\right) = \frac{41}{10}\pi$

4. (a) $y = x^2 \Rightarrow 1 + (y')^2 = 1 + 4x^2 \Rightarrow$

$$S = \int_0^1 2\pi x\sqrt{1+4x^2}\,dx = \int_1^5 \tfrac{\pi}{4}\sqrt{u}\,du \quad [u = 1+4x^2] \quad = \tfrac{\pi}{6}\left[u^{3/2}\right]_1^5 = \tfrac{\pi}{6}\left(5^{3/2} - 1\right)$$

(b) $y = x^2 \Rightarrow 1 + (y')^2 = 1 + 4x^2$. So

$$S = 2\pi\int_0^1 x^2\sqrt{1+4x^2}\,dx = 2\pi\int_0^2 \tfrac{1}{4}u^2\sqrt{1+u^2}\,\tfrac{1}{2}\,du \quad [u = 2x] \quad = \tfrac{\pi}{4}\int_0^2 u^2\sqrt{1+u^2}\,du$$

$$= \tfrac{\pi}{4}\left[\tfrac{1}{8}u(1+2u^2)\sqrt{1+u^2} - \tfrac{1}{8}\ln\left|u + \sqrt{1+u^2}\right|\right]_0^2 \quad [u = \tan\theta \text{ or use Formula 22}]$$

$$= \tfrac{\pi}{4}\left[\tfrac{1}{4}(9)\sqrt{5} - \tfrac{1}{8}\ln\left(2+\sqrt{5}\right) - 0\right] = \tfrac{\pi}{32}\left[18\sqrt{5} - \ln\left(2+\sqrt{5}\right)\right]$$

5. $y = e^{-x^2} \Rightarrow dy/dx = -2xe^{-x^2} \Rightarrow 1 + (dy/dx)^2 = 1 + 4x^2e^{-2x^2}$. Let $f(x) = \sqrt{1+4x^2e^{-2x^2}}$. Then

$$L = \int_0^3 f(x)\,dx \approx S_6 = \frac{(3-0)/6}{3}\left[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)\right] \approx 3.292287$$

6. $S = \int_0^3 2\pi y\,ds = \int_0^3 2\pi e^{-x^2}\sqrt{1+4x^2e^{-2x^2}}\,dx$. Let $g(x) = 2\pi e^{-x^2}\sqrt{1+4x^2e^{-2x^2}}$. Then

$$S = \int_0^3 g(x)\,dx \approx S_6 = \frac{(3-0)/6}{3}\left[g(0) + 4g(0.5) + 2g(1) + 4g(1.5) + 2g(2) + 4g(2.5) + g(3)\right] \approx 6.648327.$$

7. $y = \int_1^x \sqrt{\sqrt{t}-1}\,dt \quad \Rightarrow \quad dy/dx = \sqrt{\sqrt{x}-1} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + \left(\sqrt{x}-1\right) = \sqrt{x}.$

Thus, $L = \int_1^{16} \sqrt{\sqrt{x}}\,dx = \int_1^{16} x^{1/4}\,dx = \frac{4}{5}\left[x^{5/4}\right]_1^{16} = \frac{4}{5}(32-1) = \frac{124}{5}.$

8. $S = \int_1^{16} 2\pi x\,ds = 2\pi \int_1^{16} x \cdot x^{1/4}\,dx = 2\pi \int_1^{16} x^{5/4}\,dx = 2\pi \cdot \frac{4}{9}\left[x^{9/4}\right]_1^{16} = \frac{8\pi}{9}(512-1) = \frac{4088}{9}\pi$

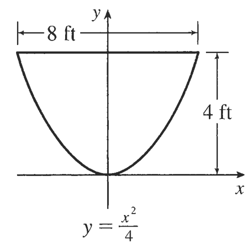
9. As in Example 1 of Section 8.3, $\dfrac{a}{2-x} = \dfrac{1}{2} \quad \Rightarrow \quad 2a = 2 - x$ and $w = 2(1.5 + a) = 3 + 2a = 3 + 2 - x = 5 - x.$

Thus, $F = \int_0^2 \rho g x(5-x)\,dx = \rho g\left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_0^2 = \rho g\left(10 - \frac{8}{3}\right) = \frac{22}{3}\delta \quad [\rho g = \delta] \quad \approx \frac{22}{3} \cdot 62.5 \approx 458$ lb.

10. $F = \int_0^4 \delta(4-y)2\left(2\sqrt{y}\right)dy = 4\delta \int_0^4 \left(4y^{1/2} - y^{3/2}\right)dy$

$= 4\delta\left[\frac{8}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^4 = 4\delta\left(\frac{64}{3} - \frac{64}{5}\right) = 256\delta\left(\frac{1}{3} - \frac{1}{5}\right)$

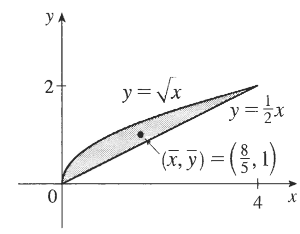
$= \frac{512}{15}\delta \approx 2133.3$ lb $\qquad [\delta \approx 62.5\ \text{lb/ft}^3]$



11. $A = \int_0^4 \left(\sqrt{x} - \frac{1}{2}x\right)dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^2\right]_0^4 = \frac{16}{3} - 4 = \frac{4}{3}$

$\overline{x} = \frac{1}{A}\int_0^4 x\left(\sqrt{x} - \frac{1}{2}x\right)dx = \frac{3}{4}\int_0^4 \left(x^{3/2} - \frac{1}{2}x^2\right)dx$

$= \frac{3}{4}\left[\frac{2}{5}x^{5/2} - \frac{1}{6}x^3\right]_0^4 = \frac{3}{4}\left(\frac{64}{5} - \frac{64}{6}\right) = \frac{3}{4}\left(\frac{64}{30}\right) = \frac{8}{5}$



$\overline{y} = \frac{1}{A}\int_0^4 \frac{1}{2}\left[\left(\sqrt{x}\right)^2 - \left(\frac{1}{2}x\right)^2\right]dx = \frac{3}{4}\int_0^4 \frac{1}{2}\left(x - \frac{1}{4}x^2\right)dx = \frac{3}{8}\left[\frac{1}{2}x^2 - \frac{1}{12}x^3\right]_0^4 = \frac{3}{8}\left(8 - \frac{16}{3}\right) = \frac{3}{8}\left(\frac{8}{3}\right) = 1$

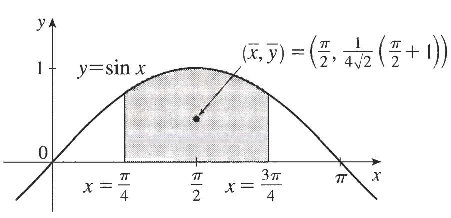
Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{8}{5}, 1\right)$.

12. From the symmetry of the region, $\overline{x} = \frac{\pi}{2}$. $\quad A = \int_{\pi/4}^{3\pi/4} \sin x\,dx = \left[-\cos x\right]_{\pi/4}^{3\pi/4} = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

$\overline{y} = \dfrac{1}{A}\displaystyle\int_{\pi/4}^{3\pi/4} \frac{1}{2}\sin^2 x\,dx = \dfrac{1}{A}\displaystyle\int_{\pi/4}^{3\pi/4} \frac{1}{4}(1 - \cos 2x)\,dx$

$= \dfrac{1}{4\sqrt{2}}\left[x - \frac{1}{2}\sin 2x\right]_{\pi/4}^{3\pi/4}$

$= \dfrac{1}{4\sqrt{2}}\left[\frac{3\pi}{4} - \frac{1}{2}(-1) - \frac{\pi}{4} + \frac{1}{2} \cdot 1\right] = \frac{1}{4\sqrt{2}}\left(\frac{\pi}{2} + 1\right)$



Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{\pi}{2}, \frac{1}{4\sqrt{2}}\left(\frac{\pi}{2} + 1\right)\right) \approx (1.57, 0.45)$.

13. An equation of the line passing through $(0, 0)$ and $(3, 2)$ is $y = \frac{2}{3}x$. $A = \frac{1}{2} \cdot 3 \cdot 2 = 3$. Therefore, using Equations 8.3.8, $\overline{x} = \frac{1}{3}\int_0^3 x\left(\frac{2}{3}x\right)dx = \frac{2}{27}\left[x^3\right]_0^3 = 2$ and $\overline{y} = \frac{1}{3}\int_0^3 \frac{1}{2}\left(\frac{2}{3}x\right)^2 dx = \frac{2}{81}\left[x^3\right]_0^3 = \frac{2}{3}$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(2, \frac{2}{3}\right)$.

14. Suppose first that the large rectangle were complete, so that its mass would be $6 \cdot 3 = 18$. Its centroid would be $\left(1, \frac{3}{2}\right)$. The mass removed from this object to create the one being studied is 3. The centroid of the cut-out piece is $\left(\frac{3}{2}, \frac{3}{2}\right)$. Therefore, for the actual lamina, whose mass is 15, $\overline{x} = \frac{18}{15}(1) - \frac{3}{15}\left(\frac{3}{2}\right) = \frac{9}{10}$, and $\overline{y} = \frac{3}{2}$, since the lamina is symmetric about the line $y = \frac{3}{2}$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{9}{10}, \frac{3}{2}\right)$.

15. The centroid of this circle, $(1, 0)$, travels a distance $2\pi(1)$ when the lamina is rotated about the y-axis. The area of the circle is $\pi(1)^2$. So by the Theorem of Pappus, $V = A(2\pi\overline{x}) = \pi(1)^2 2\pi(1) = 2\pi^2$.

16. The semicircular region has an area of $\frac{1}{2}\pi r^2$, and sweeps out a sphere of radius r when rotated about the x-axis. $\overline{x} = 0$ because of symmetry about the line $x = 0$. And by the Theorem of Pappus, $V = A(2\pi\overline{y}) \Rightarrow \frac{4}{3}\pi r^3 = \frac{1}{2}\pi r^2(2\pi\overline{y}) \Rightarrow \overline{y} = \frac{4}{3\pi}r$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{4}{3\pi}r\right)$.

17. $x = 100 \Rightarrow P = 2000 - 0.1(100) - 0.01(100)^2 = 1890$

$$\text{Consumer surplus} = \int_0^{100} [p(x) - P]\,dx = \int_0^{100} \left(2000 - 0.1x - 0.01x^2 - 1890\right) dx$$
$$= \left[110x - 0.05x^2 - \frac{0.01}{3}x^3\right]_0^{100} = 11{,}000 - 500 - \frac{10{,}000}{3} \approx \$7166.67$$

18. $$\int_0^{24} c(t)\,dt \approx S_{12} = \frac{24-0}{12 \cdot 3}[1(0) + 4(1.9) + 2(3.3) + 4(5.1) + 2(7.6) + 4(7.1) + 2(5.8) + 4(4.7) + 2(3.3) + 4(2.1) + 2(1.1) + 4(0.5) + 1(0)]$$
$$= \frac{2}{3}(127.8) = 85.2 \text{ mg} \cdot \text{s/L}$$

Therefore, $F \approx A/85.2 = 6/85.2 \approx 0.0704$ L/s or 4.225 L/min.

19. $f(x) = \begin{cases} \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right) & \text{if } 0 \le x \le 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$

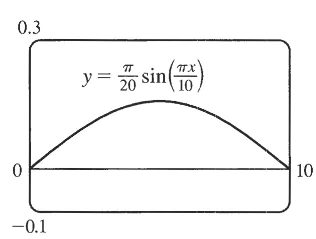
(a) $f(x) \ge 0$ for all real numbers x and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right) dx = \frac{\pi}{20} \cdot \frac{10}{\pi}\left[-\cos\left(\frac{\pi}{10}x\right)\right]_0^{10} = \frac{1}{2}(-\cos\pi + \cos 0) = \frac{1}{2}(1+1) = 1$$

Therefore, f is a probability density function.

(b) $$P(X < 4) = \int_{-\infty}^{4} f(x)\,dx = \int_0^4 \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right) dx = \frac{1}{2}\left[-\cos\left(\frac{\pi}{10}x\right)\right]_0^4 = \frac{1}{2}\left(-\cos\frac{2\pi}{5} + \cos 0\right)$$
$$\approx \frac{1}{2}(-0.309017 + 1) \approx 0.3455$$

(c) $$\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{10} \frac{\pi}{20}x\sin\left(\frac{\pi}{10}x\right) dx$$
$$= \int_0^{\pi} \frac{\pi}{20} \cdot \frac{10}{\pi}u(\sin u)\left(\frac{10}{\pi}\right) du \qquad \left[u = \frac{\pi}{10}x,\ du = \frac{\pi}{10}\,dx\right]$$
$$= \frac{5}{\pi}\int_0^{\pi} u\sin u\,du \overset{82}{=} \frac{5}{\pi}[\sin u - u\cos u]_0^{\pi} = \frac{5}{\pi}[0 - \pi(-1)] = 5$$



This answer is expected because the graph of f is symmetric about the line $x = 5$.

20. $P(250 \leq X \leq 280) = \displaystyle\int_{250}^{280} \frac{1}{15\sqrt{2\pi}} \exp\left(\frac{-(x-268)^2}{2 \cdot 15^2}\right) dx \approx 0.673$. Thus, the percentage of pregnancies that last between 250 and 280 days is about 67.3%.

21. (a) The probability density function is $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{8}e^{-t/8} & \text{if } t \geq 0 \end{cases}$

$P(0 \leq X \leq 3) = \int_0^3 \frac{1}{8}e^{-t/8}\,dt = \left[-e^{-t/8}\right]_0^3 = -e^{-3/8} + 1 \approx 0.3127$

(b) $P(X > 10) = \int_{10}^{\infty} \frac{1}{8}e^{-t/8}\,dt = \lim\limits_{x\to\infty} \left[-e^{-t/8}\right]_{10}^{x} = \lim\limits_{x\to\infty} (-e^{-x/8} + e^{-10/8}) = 0 + e^{-5/4} \approx 0.2865$

(c) We need to find m such that $P(X \geq m) = \frac{1}{2} \Rightarrow \int_m^{\infty} \frac{1}{8}e^{-t/8}\,dt = \frac{1}{2} \Rightarrow \lim\limits_{x\to\infty} \left[-e^{-t/8}\right]_m^x = \frac{1}{2} \Rightarrow$

$\lim\limits_{x\to\infty} (-e^{-x/8} + e^{-m/8}) = \frac{1}{2} \Rightarrow e^{-m/8} = \frac{1}{2} \Rightarrow -m/8 = \ln\frac{1}{2} \Rightarrow m = -8\ln\frac{1}{2} = 8\ln 2 \approx 5.55$ minutes.